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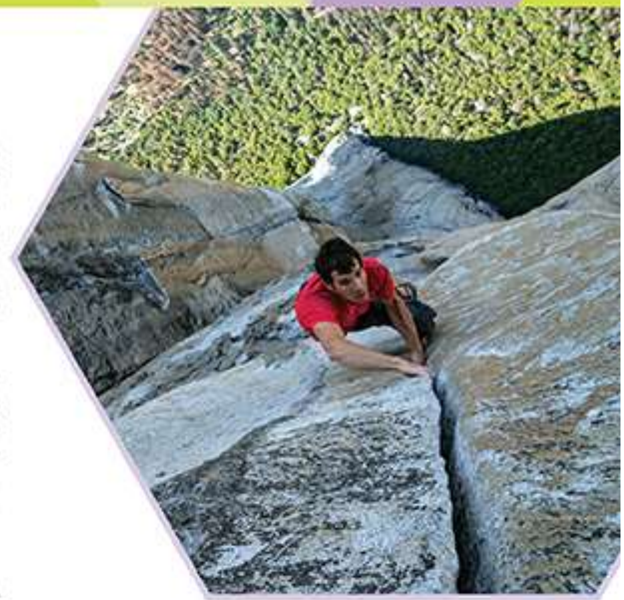
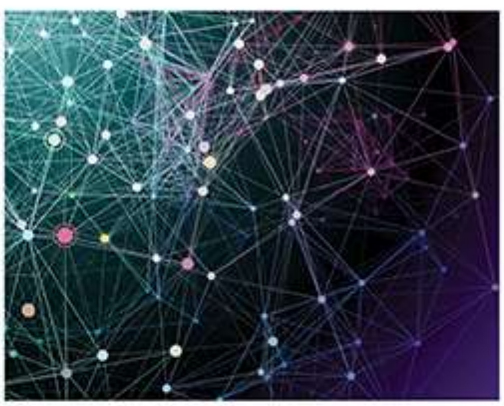
Reaching the Pinnacle

In 1958 the first recorded ascent of the 3000 foot rock face of El Capitan in Yosemite National Park took the team of three climbers 47 days. The entire project took eighteen months as the climbers established multiple camps along the route—all connected by ropes. In 2017 Alex Honnold, using only a small bag of chalk strapped to his waist, climbed El Capitan free solo in under four hours.



Honnold did not accomplish this feat on a whim. He practiced climbing from a young age, noting that others had more natural ability, but he just kept on working. He trained hard (hanging by just his fingertips up to an hour every day), carefully plotted his route, and practiced particularly challenging parts of the climb by repeatedly rappelling down and climbing up.

In Precalculus you will hone and combine skills you have developed in algebra and geometry. For many of you, this course will be the final preparation you need before tackling a college-level calculus course. For others, it may be your final math class before you turn your focus to other areas. Either way, if you invest the necessary effort, you will expand both your skill in solving problems and your understanding of mathematics—its history, current use, and relevance from a biblical perspective. In the current Information Age with its data-driven algorithms, these skills are essential to understanding complex issues in health care, law enforcement, social media, space flight, and myriad other areas.



Over the next few pages you will be introduced to some of the special features contained in this book. The text integrates answers to critical questions regarding the genesis of mathematical ideas, modern applications of these ideas, and the importance of these matters in today's world. Frequent technology instruction, tips, and step-by-step examples will assist you in your journey. In addition, margin notes in each section provide interesting bits of information that directly relate to the development or application of ideas in the accompanying text.

The Historical Connection, Data Analysis, and Biblical Perspective of Mathematics features found in each chapter provide context, application, and meaning to the mathematical ideas as they are introduced and will improve comprehension and retention. Woven together, these three features present a strong biblical worldview of mathematics. The Historical Connection features provide more than a timeline of events; they explore the factors leading to the development of particular mathematical ideas and the reasons why they were discovered at a particular time and place. The Data Analysis features offer an opportunity to work on real-world problems using the tools learned in that chapter. While the entire text is written from a biblical worldview that rejects the moral neutrality of mathematics, particular worldview topics are addressed in detail in the Biblical Perspective of Mathematics features.

We hope you are excited about this course! The material may be challenging, but the accomplishments are well worth the effort. Take advantage of the many features that are included in the text to help you along the way. Our goal is that you find this book useful, rigorous, and consistent with a biblical worldview.

4 Trigonometric Functions

HISTORICAL CONNECTION

Trigonometry, born out of an effort to understand movement in the "celestial sphere," initially had as much to do with circles and spheres as it did with triangles.

BIBLICAL PERSPECTIVE OF MATHEMATICS

The development of abstract mathematics has allowed us to explore the vastness of God's creation. Stanley L. Jaki notes in *The Relevance of Physics* that "the science of trigonometry was in a sense a precursor of the telescope. It brought faraway objects within the compass of measurement and first made it possible for man to penetrate in a quantitative manner the far reaches of space."

DATA ANALYSIS

Regularly recurring (cyclic) patterns are found in abundance in natural and manmade systems. Since trigonometric functions are cyclical, they are used to model many practical applications, including the occurrence of sunspots.

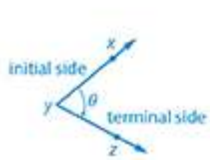
4.1 Angle Measure and Arc Length

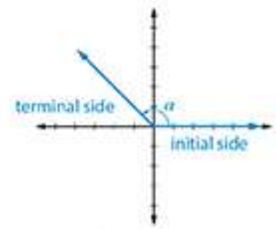
This statue on Nepean Point in Ottawa depicts the French explorer Samuel de Champlain holding an astrolabe, which would have helped him navigate by using the sun's angle of elevation above the horizon.

After completing this section, you will be able to

- draw an angle in standard position.
- convert between degrees, minutes, and seconds; decimal degrees; and radian measures of angles.
- identify coterminal angles.
- calculate arc length and sector area using angle measure and radius length.
- calculate angular and linear speeds.

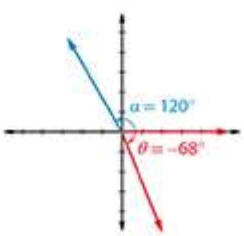
The term *trigonometry* is derived from Greek words meaning "triangle measure." Ancient civilizations used the relationships between the sides and angles of right triangles in surveying, navigation, engineering, and astronomy. Trigonometry was then expanded to explain circular and harmonic phenomena, including orbits, vibrating strings, pendulums, and electrical current.





In geometry, an angle is often defined as the union of two rays with a common endpoint, or *vertex*. An angle can also be viewed as the sweep of a ray moving from an initial position to a final position, similar to the sweep of a clock's hand in a given period of time. An angle is in *standard position* when it is placed on the coordinate plane with its vertex at the origin and the initial position of the ray aligned with the positive x-axis. The final position of the sweeping ray is the *terminal side* of the angle.

An angle is often named by its vertex, such as $\angle Y$. When more than one angle has its vertex at Y, three letters are used to avoid confusion, as in $\angle XYZ$. In this text, $m\angle Y$ is frequently notated simply as Y. Lowercase Greek letters such as α (alpha), β (beta), or θ (theta) are also used to name an angle or indicate its measure.

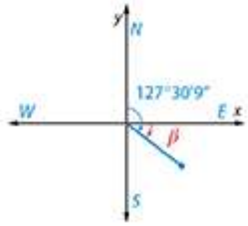


When an angle is viewed as a rotation, its measure includes both the direction and amount of rotation. In mathematics, a counterclockwise rotation is described as a *positive angle* and a clockwise rotation is described as a *negative angle*. The amount of rotation can be measured in *degrees*, where 1° represents $\frac{1}{360}$ of a circle. In the illustration, $\alpha = 120°$ and $\theta = -68°$. Degrees can be further divided using decimal degrees (DD) or degrees, minutes, and seconds (DMS), in which a *minute* (1′) is $\frac{1}{60}$ of a degree and a *second* (1″) is $\frac{1}{60}$ of a minute (or $\frac{1}{3600}$ of a degree). In navigation, angles representing bearings are measured from due north where clockwise rotations are positive.

Example 1 Describing a Navigational Bearing

Represent a ship's bearing, given in DMS form as 127°30′9″, as a standard position angle in DD form.

Answer



1. Sketch the ship's bearing.

$$127^\circ + 30'\left(\frac{1^\circ}{60'}\right) + 9''\left(\frac{1^\circ}{3600''}\right)$$
$$\approx 127^\circ + 0.5^\circ + 0.0025^\circ$$
$$\approx 127.5025^\circ$$

2. Convert DMS form to DD form.

$$127.5025^\circ - 90^\circ = 37.5025^\circ$$
$$\beta = -37.5025^\circ$$

3. Express the terminal ray as a negative rotation from the positive *x*-axis.

Check

$$0.5025^\circ\left(\frac{60'}{1^\circ}\right) = 30.15'$$

4. Convert the decimal portion of the degrees to minutes.

$$30' + 0.15'\left(\frac{60''}{1'}\right) = 30'\,9''$$

5. Convert the decimal portion of the minutes to seconds.

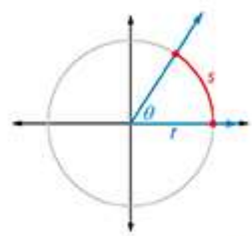
SKILL ✓ EXERCISE 9

The DMS divisions of a circle are rooted in the Babylonian sexagesimal (base 60) number system and the ease of dividing a circle's circumference into equal arcs with 6 equilateral triangles. The fact that a year is approximately 360 days may also have been an influencing factor.

While degrees work well for navigation and construction, many applications involving trigonometric functions require the domain to be real numbers. Using *radian measure* allows both the domain and the range to be expressed with real numbers having similar units.

DEFINITIONS

The **radian measure** of an angle θ is the ratio of its intercepted arc's length s to the circle's radius r: $\theta = \frac{s}{r}$. A central angle that intercepts an arc whose length is the same as the circle's radius has a measure of 1 **radian**.



Note that a radian measure is a real number with no units since it is a ratio of two lengths. The relationship between radian and degree measures can be derived by considering the arc length of one complete rotation of 360°: $s = 2\pi r$, the circumference of a circle.

$$\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi \text{ radians} = 360^\circ$$
$$\pi \text{ radians} = 180^\circ$$

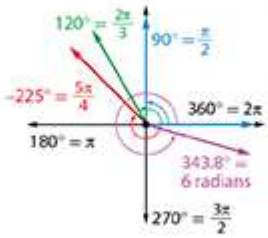
TIP

This text occasionally includes the word *radians* with a radian measure for clarity.

This relationship produces two unit multipliers that can be used to convert between radians and degrees.

$$\frac{\pi \text{ radians}}{180^\circ} = 1 \text{ and } \frac{180^\circ}{\pi \text{ radians}} = 1$$

Example 2 Converting between Degree and Radian Measures



Convert each degree measure to radians and each radian measure to degrees.

a. $90°$ **b.** $-225°$ **c.** $\frac{2\pi}{3}$ **d.** 6

Answer

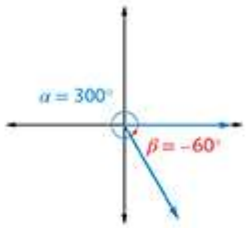
a. $90°\left(\frac{\pi \text{ radians}}{180°}\right) = \frac{\pi}{2} \approx 1.57$

b. $-225°\left(\frac{\pi \text{ radians}}{180°}\right) = -\frac{5\pi}{4} \approx -3.93$

c. $\frac{2\pi}{3}\left(\frac{180°}{\pi \text{ radians}}\right) = 120°$

d. $6\left(\frac{180°}{\pi \text{ radians}}\right) = \frac{1080}{\pi} \approx 343.8°$

SKILL ✓ EXERCISE 11

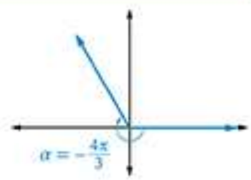


The terminal side of an *acute* angle (radian measure $0 < \theta < \frac{\pi}{2}$) lies in Quadrant I, and the terminal side of an *obtuse* angle ($\frac{\pi}{2} < \theta < \pi$) lies in Quadrant II. The terminal side of a *quadrantal angle* ($\theta = n\frac{\pi}{2}, n \in \mathbb{Z}$) lies on one of the axes.

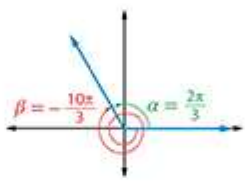
Two angles in standard position with the same terminal ray are called *coterminal angles*. Since the dynamic view of angles allows a ray to sweep in either direction or to sweep through one or more complete circles before taking its terminal position, any angle has an infinite number of coterminal angles. Coterminal angles can be found by adding or subtracting multiples of 360° or 2π radians. For example, a 300° angle is coterminal with angles of −60°, 660°, and −420°.

Example 3 Finding Coterminal Angles

Write an expression for all angles coterminal with $\theta = -\frac{4\pi}{3}$. Then specify one positive and one negative coterminal angle.



Answer



$-\frac{4\pi}{3} + n2\pi, n \in \mathbb{N}$	1. Coterminal angles can be found by adding and subtracting multiples of 2π.
$\alpha = -\frac{4\pi}{3} + (1)2\pi = \frac{2\pi}{3}$	2. Let $n = 1$ and add 2π to find a positive coterminal angle.
$\beta = -\frac{4\pi}{3} - (1)2\pi = -\frac{10\pi}{3}$	3. Let $n = -1$ and subtract 2π to find a negative coterminal angle.

SKILL ✓ EXERCISE 21

Solving the definition of radian measure, $\theta = \frac{s}{r}$, for r leads directly to a formula for *arc length*.

ARC LENGTH FORMULA

A central angle of radian measure θ in a circle with radius length r intercepts an arc with length $s = r\theta$.

Note that when $r = 1$, the arc length is equal to the radian measure of the central angle. When the central angle is given in degrees, convert to radian measure before applying the arc length formula.

Example 4 Finding Arc Length

A nautical mile has historically been defined as 1 minute of latitude at the earth's equator. Determine the relationship between a (statute) mile and a nautical mile (nm), given that the earth's average radius at the equator is 3959 mi.

Answer

$\theta = \frac{1°}{60}\left(\frac{\pi \text{ radians}}{180°}\right) = \frac{\pi}{10{,}800}$

1. Convert 1 minute to radian measure.

$s = r\theta = (3959 \text{ mi})\left(\frac{\pi}{10{,}800}\right)$
$\approx 1.15 \text{ mi}$

2. Apply the arc length formula.

$\therefore 1 \text{ nm} \approx 1.15 \text{ mi}$

SKILL ✓ EXERCISE 37

Creating an accurate calendar during Old Testament times was a difficult task requiring a thorough understanding of both mathematics and astronomy. Months based on cycles of the moon were about 10 days short of a solar year, so an extra month was periodically added to keep the months aligned with the seasons. Even with leap years, our current model must be adjusted every 400 years.

The study of *uniform circular motion* analyzes the linear and angular speeds of an object moving in a circular path at a constant speed. The *angular speed*, usually denoted with the lowercase Greek letter omega (ω), is the ratio of the angle of rotation (in radians) per unit of time: $\omega = \frac{\theta}{t}$. Angular speed stated in terms of revolutions can be converted to radians using the unit multiplier $\frac{2\pi \text{ radians}}{1 \text{ revolution}}$. The *linear speed* is the ratio of arc length, or distance traveled, per unit of time: $v = \frac{s}{t}$. Since $s = r\theta$, $v = \frac{r\theta}{t} = r\left(\frac{\theta}{t}\right) = r\omega$.

UNIFORM CIRCULAR MOTION FORMULAS

If θ is the radian measure of the angle of rotation,
then the angular speed is $\omega = \frac{\theta}{t}$ and the linear speed is $v = \frac{s}{t} = r\omega$.

Example 5 Analyzing Uniform Circular Motion

Karyn is playing an LP vinyl record at $33\frac{1}{3}$ rpm (revolutions per minute) on her vintage record player. Determine the angular speed of the record (in radians per second). Then find the linear speed (in inches per second) of a point at the edge of the record (6 in. from the center) and the linear speed of a point at the edge of its label (2 in. from the center).

Answer

$\omega = \frac{100 \text{ revolutions}}{3 \text{ min}}\left(\frac{2\pi \text{ radians}}{1 \text{ revolution}}\right)\left(\frac{1 \text{ min}}{60 \text{ sec}}\right)$
$= \frac{10\pi \text{ radians}}{9 \text{ sec}} \approx 3.49 \text{ radians/sec}$

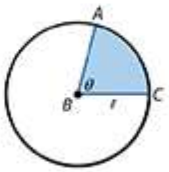
1. Convert the angular speed from revolutions per minute to radians per second.

$v_6 = r\omega = (6 \text{ in.})\left(\frac{10\pi}{9 \text{ sec}}\right)$
$= \frac{20\pi \text{ in.}}{3 \text{ sec}} \approx 20.94 \text{ in./sec}$
$v_2 = (2 \text{ in.})\left(\frac{10\pi}{9 \text{ sec}}\right) = \frac{20\pi \text{ in.}}{9 \text{ sec}} \approx 6.98 \text{ in./sec}$

2. Substitute into $v = r\omega$ to find each linear speed.
Since the radian measure is a ratio of two lengths, it is a real number and the label is not needed.

SKILL ✓ EXERCISE 33

Radian measure provides a convenient formula for the area of a *sector* of a circle, the region bounded by two radii and the intercepted arc. Recall from GEOMETRY that the ratio of the area of a sector and the circle's area is equal to ratio of the central angle and a complete rotation: $\frac{A_{sector}}{\pi r^2} = \frac{\theta}{2\pi}$. Solving for the area of the sector, $A_{sector} = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \frac{1}{2}r^2\theta$.



Example 6 Finding the Area of a Sector

If the shaded sector has an area of $24\pi\ m^2$ and $BA = 12$ m, find the radian measure and the degree measure of $\angle ABC$.

Answer

$A_{sector} = \frac{1}{2}r^2\theta$ $\theta = \frac{2A_{sector}}{r^2}$	1. Solve the formula for the area of a sector for θ, the radian measure of the angle.
$\theta = \frac{2(24\pi\ m^2)}{(12\ m)^2} = \frac{48\pi\ m^2}{144\ m^2} = \frac{\pi}{3}$	2. Substitute and simplify to express θ in radians.
$\frac{\pi}{3}$ radians $\left(\frac{180°}{\pi \text{ radians}}\right) = 60°$	3. Convert radians to degrees.

SKILL ✓ EXERCISE 31

A. Exercises

1. Sketch a standard position angle with each degree measure.

a. 45° **b.** −30°

c. −225° **d.** 500°

2. Sketch a standard position angle with each radian measure.

a. $\frac{5\pi}{4}$ **b.** $-\pi$

c. $\frac{-4\pi}{3}$ **d.** $\frac{15\pi}{4}$

Convert to degrees, minutes, and seconds (DMS).

3. −2.87° **4.** 110.51° **5.** 48.362°

Convert to decimal degrees (DD).

6. −58°54′ **7.** 98°45′43.2″ **8.** 135°35′38.4″

Convert each degree measure to radian measure. Express the answer in terms of π and as a decimal approximation rounded to the nearest hundredth.

9. 35° **10.** −40°

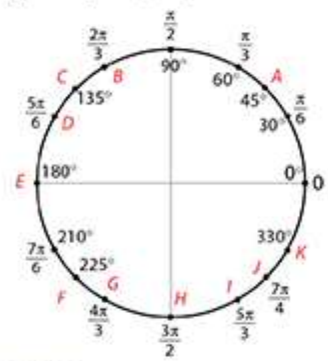
11. 1080° **12.** 154.5°

Convert each radian measure to degrees. Round answers to the nearest tenth of a degree if necessary.

13. $\frac{\pi}{5}$ **14.** $-\frac{7\pi}{2}$

15. $\frac{\pi}{12}$ **16.** 5

17. Identify the missing degree or radian measure of the standard position angle whose terminal side passes through each point (A–K).



B. Exercises

18. How many degrees (to the nearest tenth) are in one radian? How many radians (to the nearest ten thousandth) are in one degree?

For each angle, write an expression for all coterminal angles. Then state one positive and one negative coterminal angle.

19. $\alpha = 148°$ **20.** $\alpha = -200°$

21. $\alpha = \frac{\pi}{4}$ **22.** $-\frac{5\pi}{3}$

23. Which of the following angles are coterminal to $\alpha = \frac{\pi}{5}$? Select all that apply.

A. $\frac{4\pi}{5}$ **B.** $\frac{11\pi}{5}$ **C.** 756° **D.** −324°

Find the missing arc length, radius measure, or angle measure (in radians).

	Arc (s)	Radius (r)	Angle (θ)
24.		8	$\frac{\pi}{4}$
25.	6π	4	
26.	$\frac{\pi}{2}$		$\frac{\pi}{6}$
27.	3π	6	

28. An air traffic controller notes that an airplane flying from Kansas City to Los Angeles at a bearing of 259°38′6″ will be landing shortly. Represent the plane's bearing as a standard position angle in decimal degrees (DD).

29. If the pendulum of a grandfather clock is 94 cm long, how far does the end of the pendulum travel as it swings through an angle of 6°? If the pendulum is lengthened by 2 cm to prevent the clock from running too fast, how much farther does the end swing through its 6° arc? Round answers to the nearest hundredth of a centimeter.

30. If a sprinkler rotates 60° and the nozzle streams water 5 ft away, find the area watered (to the nearest tenth of a square foot).

31. A 24 in. diameter prize wheel is divided into 16 equal sectors. Find the arc length and the area of one sector. Round answers to the nearest tenth.

32. A compact circular saw operates at a maximum speed of 3500 rpm and has a blade with a diameter of 4.5 in. Find the angular speed of the blade (in radians per second) and the linear speed (in inches per second) for a tip of the sawblade. Round answers to the nearest tenth.

33. A floor buffer with a 20 in. radius operates at 1500 rpm. Find the angular speed of the buffer (in radians per second) and the difference of the linear speeds (in miles per hour) for a point on the outer edge of the buffer pad and a point on the edge of the 3 in. diameter hole in the center of the pad. Round answers to the nearest tenth.

34. Show algebraically that every angle coterminal with 2π is an even multiple of π.

35. Show algebraically that every angle coterminal with π is an odd multiple of π.

36. Write an equation expressing the difference of any two coterminal angles α and θ.

C. Exercises

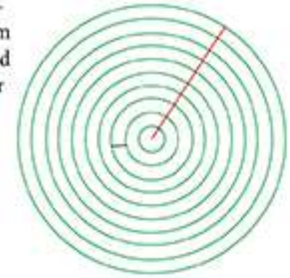
37. Minneapolis, Minnesota, is located at 44°59′ N and Springfield, Missouri, is located at 37°13′ N on the same longitudinal line. Estimate the distance between the cities (to the nearest mile), using 3959 mi as the radius of the earth.

38. Samuel is making a 3D pie chart with a 3 ft diameter for a presentation and plans to cover each slice with fabric. If the pie chart is 6 in. thick, how much fabric (to the nearest tenth of a square foot) will be used to cover the top and the sides of the slice that represents 40% of the spending?

39. An amusement park is planning the *Centrifuge*, a new ride that presses passengers against the wall of a circular cylinder as it spins at 24 rpm.

- **a.** Determine the angular speed of the ride (in radians per second).
- **b.** Find the linear speed (in miles per hour) of a person on the wall for rides with diameters of 45 ft and 48 ft.

40. A 1300 ft long center-pivot irrigation system with 10 equally spaced trusses takes 72 hr for each rotation over a circular field.



- **a.** Find the distance that the third irrigation truss from the center travels in 9 hr.
- **b.** How many acres are irrigated between the third and outermost trusses in 9 hr? (1 acre = 43,560 ft^2)
- **c.** Find the angular speed (in radians per hour) of the irrigation system.
- **d.** Find the linear speed (in feet per hour) of the first truss and the ninth truss.