Solution

(a) Area of cross-section of the swimming pool

$= \frac{1}{2} \times (1 + 3 - 0.75) \times 100$

$= 162.5 \text{ m}^2$

$\therefore$ volume of water in the pool $= 162.5 \times 40$

$= 6{,}500 \text{ m}^3$

(b) Volume of water to be poured $= 100 \times 40 \times (1.75 - 1)$

$= 4{,}000 \times 0.75$

$= 3{,}000 \text{ m}^3$

(c) Total volume of fully filled pool $= 3{,}000 + 6{,}500$

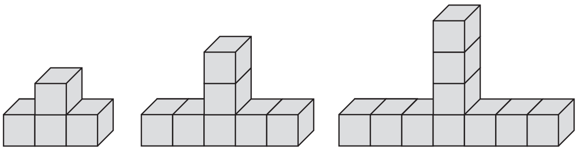
$= 9{,}500 \text{ m}^3$

Time taken to drain fully filled pool $= \frac{9{,}500}{10} \times \frac{1}{2}$

$= 475$ min

$= 7$ hr 55 min

25. The diagram shows a sequence of figures formed by stacking solid cubes together. The volume of each cube is 1 cm^3.



$n = 1$ $\quad$ $n = 2$ $\quad$ $n = 3$

Let the volume and total surface area of the cubes in the nth figure be $V_n \text{ cm}^3$ and $A_n \text{ cm}^2$ respectively.

(a) Complete the following table.

n	1	2	3
V_n	4		
A_n		30	42

(b) Find an expression for
- **(i)** V_n,
- **(ii)** A_n.

(c) Hence, show that
- **(i)** the total surface area of the cubes in any figure is divisible by 6,
- **(ii)** $4V_n = A_n - 2$.

(d) The total surface area of all the cubes in the pth figure is 102 cm^2. Find
- **(i)** the corresponding volume of the cubes.
- **(ii)** the value of p.

Solution

(a)

n	1	2	3
V_n	4	7	10
A_n	18	30	42

(b) **(i)** $V_n = 3n + 1$

(ii) $A_n = 12n + 6$

(c) **(i)** $A_n = 12n + 6$

$= 6(2n + 1)$

$\therefore$ the total surface area of the cubes in any figure is divisible by 6.

(ii) $4V_n = 4(3n + 1)$

$= 12n + 4$

$= 12n + 6 - 2$

$= A_n - 2$ (shown)

(d) **(i)** $4V_p = A_p - 2$

$= 102 - 2$

$= 100$

$V_p = 25$

$\therefore$ Corresponding volume $= 25 \text{ cm}^2$

(ii) $3p + 1 = V_p$

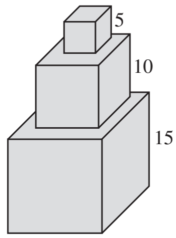
$= 25$

$3p = 24$

$p = 8$

Enrichment

26. In the diagram, three cubical building blocks are stacked up on a table. The lengths of the sides of the blocks are 5 cm, 10 cm, and 15 cm respectively.



(a) Find the total area of the exposed surfaces of the stack, excluding the contact surface with the table.

(b) If a cylinder of height 30 cm has volume equal to the total volume of the blocks, find the base radius of the cylinder.

Solution

(a) Total area of the exposed top surfaces of the cubes

= Area of the top face of the largest cube

$= 15 \times 15$

$= 225 \text{ cm}^2$

Total area of the exposed lateral surfaces of the cubes

$= 4 \times (5 \times 5 + 10 \times 10 + 15 \times 15)$

$= 1{,}400 \text{ cm}^2$

$\therefore$ required total area $= 225 + 1{,}400$

$= 1{,}625 \text{ cm}^2$

(b) Let r cm be the base radius of the cylinder.

$\pi r^2 \times 30 = 5^3 + 10^3 + 15^3$

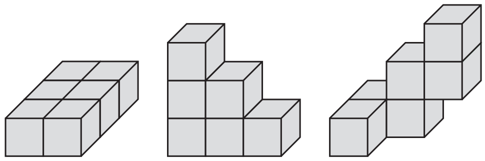
$30\pi r^2 = 4{,}500$

$r = \sqrt{\frac{150}{\pi}}$

$= 6.91$ (correct to 2 d.p.)

The base radius of the cylinder is 6.91 cm.

27. Six cubes of side 1 cm are glued together to form a solid. Three possible solids P, Q, and R are shown below.



P $\quad$ Q $\quad$ R

(a) Determine the total surface area of solid
(i) P, **(ii)** Q, **(iii)** R.

(b) Form a solid with the least total surface area.

(c) Form a solid with the greatest total surface area.

Solution

(a) **(i)** Before the cubes are glued together, number of exposed faces $= 6 \times 6 = 36$

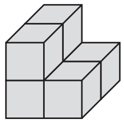
Hence, the total surface area is 36 cm^2.

In solid P, 7 pairs of faces are glued together.
$7 \times 2 = 14$ faces are no longer exposed.
$\therefore$ total surface area of solid $P = 36 - 14$
$= 22 \text{ cm}^2$

(ii) In solid Q, 6 pairs of faces are glued together.
$\therefore$ total surface area of solid $Q = 36 - (6 \times 2)$
$= 24 \text{ cm}^2$

(iii) In solid R, 5 pairs of faces are glued together.
$\therefore$ total surface area of solid $R = 36 - (5 \times 2)$
$= 26 \text{ cm}^2$

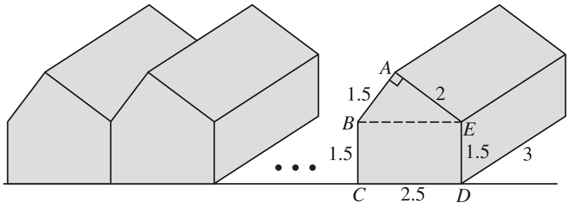
(b) The least total surface area of the solid is 22 cm^2.
The solid formed may be P or S as shown.



S

(c) The greatest total surface area of the solid is 26 cm^2.
The solid formed may be R or T as shown.

28.



A developer builds a row of identical semi-detached huts along a beach as shown in the diagram above. $ABCDE$ is the cross-section of a hut. $\triangle ABE$ is a right-angled triangle with $AB = 1.5$ m, $AE = 2$ m, and $m\angle BAE = 90°$. $BCDE$ is a rectangle with $CD = 2.5$ m and $BC = 1.5$ m. The length of each hut is 3 m. The thickness of each side wall is 30 cm.

(a) Find the total surface area of each hut, excluding the floor.
(b) Find the volume of space of each hut (ignore the thickness of the walls).
(c) If n huts are in a row, find, in terms of n,
(i) the total roof area,
(ii) the total volume of the side walls.

Solution

(a) Area of $ABCDE = \frac{1}{2} \times 1.5 \times 2 + 1.5 \times 2.5$
$= 5.25 \text{ m}^2$
Area of roof $= (AB + AE) \times 3$
$= (1.5 + 2) \times 3$
$= 10.5 \text{ m}^2$
Area of each wall $= 1.5 \times 3$
$= 4.5 \text{ m}^2$
Total surface area of a hut
$= 5.25 \times 2 + 10.5 + 4.5 \times 2$
$= 30 \text{ m}^2$

(b) Volume of space of a hut $= 5.25 \times 3$
$= 15.75 \text{ m}^3$

(c) **(i)** Total roof area $= 10.5 \times n$
$= 10.5n \text{ m}^2$

(ii) There are $(n + 1)$ side walls for n huts.
Total volume of the side walls
$= 4.5 \times 0.30 \times (n + 1)$
$= 1.35(n + 1) \text{ m}^3$

Chapter 14 Proportions

Basic Practice

1. Express each of the following scales in the form $1 : r$.

(a) 1 in. : 5 ft	**(b)** 1 in. : 4 yd
(c) 5 ft : 1 mi	**(d)** 32 yd : 1 mi
(e) 3 cm : 600 m	**(f)** 4 cm : 500 m
(g) 8 cm : 3.2 km	**(h)** 0.2 cm : 0.04 km

Solution

(a) 1 in. : 5 ft
= 1 in. : (5 × 12) in.
= 1 : 60

(b) 1 in. : 4 yd
= 1 in. : (4 × 3) ft
= 1 in. : (4 × 3 × 12) in.
= 1 : 144

(c) 5 ft : 1 mi
= 5 ft : 1,760 yd
= 5 ft : (1,760 × 3) ft
= 1 : 1,056

(d) 32 yd : 1 mi
= 32 yd : 1,760 yd
= 1 : 55

(e) 3 cm : 600 m
= 3 cm : 60,000 cm
= 1 : 20,000

(f) 4 cm : 500 m
= 4 cm : 50,000 cm
= 1 : 12,500

(g) 8 cm : 3.2 km
= 1 cm : 400 m
= 1 cm : 40,000 cm
= 1 : 40,000

(h) 0.2 cm : 0.04 km
= 1 cm : 0.2 km
= 1 cm : 200 m
= 1 cm : 20,000 cm
= 1 : 20,000

2. The scale of a map is $\frac{1}{250{,}000}$. Find the actual distance, in km, for each of the following distances on the map.

(a) 1 cm	**(b)** 6 cm
(c) 0.8 cm	**(d)** 2.5 cm
(e) 30 mm	**(f)** 45 mm

Solution

(a) Actual distance = 1 × 250,000 cm
= 250,000 cm
= 2,500 m
= 2.5 km

(b) Actual distance = 6 × 250,000 cm
= 1,500,000 cm
= 15,000 m
= 15 km

(c) Actual distance = 0.8 × 250,000 cm
= 200,000 cm
= 2,000 m
= 2 km

(d) Actual distance = 2.5 × 250,000 cm
= 625,000 cm
= 6,250 m
= 6.25 km

Solution

(a) Sample space, S = { (T, H, H), (H, H, T), (H, T, H), (T, T, H), (T, H, T), (H, T, T), (T, T, T), (H, H, H)}

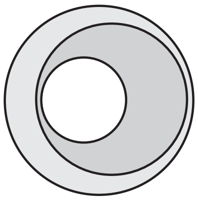
(b) (i) P(all heads) $= \frac{1}{8}$

(ii) P(2 heads and a tail) $= \frac{3}{8}$

(iii) P($\leqslant$ 1 head) $= \frac{4}{8} = \frac{1}{2}$

(iv) P(no heads) = P(all tails) $= \frac{1}{8}$

29. In the diagram, the diameters of the 3 circles are in the ratio 3 : 7 : 8. A point within the diagram is selected at random. Find the probability of selecting a point in



(a) the smallest circle,
(b) the shaded region,
(c) the space between the biggest circle and the middle circle.

Solution

(a) P(point in the smallest circle) $= \left(\frac{3}{8}\right)^2 = \frac{9}{64}$

(b) P(point in the shaded region) $= \frac{7^2 - 3^2}{8^2} = \frac{40}{64} = \frac{5}{8}$

(c) P(point in the space between the biggest and the middle circles) $= \frac{8^2 - 7^2}{8^2} = \frac{15}{64}$

Challenging Practice

30. Let ξ be the set of employees in a company,
P = {employees in the company who earn more than \$2,000 a month}
and Q = {employees in the company who earn at least \$3,000 a month}.
(a) Describe the sets P' and Q'.
(b) Describe using '$\subset$', the relationship between
(i) P and Q,
(ii) P' and Q'.

Solution

(a) P' is the set of employees in a company who earn at most \$2,000 a month.
Q' is the set of employees in a company who earn less than \$3,000 a month.

(b) (i) $Q \subset P$
(ii) $P' \subset Q'$

31. The frequency table below shows the number of dogs owned by a group of children.

Number of dogs	0	1	2	3	4	5
Number of children	7	10	5	5	x	1

(a) The mean number of dogs owned by each child is 1.6. Form an equation in x and solve it.
(b) Hence, find the number of children in the group.
(c) A child is randomly selected. Find the probability of selecting a child with more than 3 dogs.
(d) A dog is randomly selected. Find the probability of selecting a dog that belongs to a child who has at most 3 dogs.

Solution

(a)
$$\frac{0 \times 7 + 1 \times 10 + 2 \times 5 + 3 \times 5 + 4 \times x + 5 \times 1}{7 + 10 + 5 + 5 + x + 1} = 1.6$$
$$\frac{10 + 10 + 15 + 4x + 5}{28 + x} = \frac{8}{5}$$
$$\frac{40 + 4x}{28 + x} = \frac{8}{5}$$
$$200 + 20x = 224 + 8x$$
$$12x = 24$$
$$x = 2$$

(b) Number of children $= 7 + 10 + 5 + 5 + 2 + 1 = 30$

(c) P(child with > 3 dogs) $= \frac{2 + 1}{30} = \frac{1}{10}$

(d) Number of dogs owned by children with $\geqslant$ 3 dogs
$= 3 \times 5 + 4 \times 2 + 5 \times 1$
$= 28$
Total number of dog $= 40 + 4 \times 2 = 48$
$\therefore$ P(dog belongs to a child with $\geqslant$ 3 dogs) $= \frac{28}{48} = \frac{7}{12}$

32. A box contains 200 buttons that are either blue or green. A button is randomly selected from the box.
(a) Find the number of each type of button if the probability of selecting a blue button is $\frac{11}{25}$.
(b) How many blue buttons must be removed from the 200 buttons so that the probability of selecting a green button will become $\frac{8}{13}$?
(c) How many blue buttons must be added to the 200 buttons so that the probability of selecting a green button will become $\frac{14}{27}$?
(d) When x blue buttons are added and x green buttons are removed from the 200 buttons, the probability of selecting either a blue or green button is the same. Find the value of x.

Solution

(a) P(blue) $= \frac{11}{25}$

$\therefore$ number of blue buttons $= \frac{11}{25} \times 200 = 88$

Number of green buttons $= 200 - 88 = 112$

(b) Let number of blue buttons to be removed be w.

$$\therefore \frac{112}{200-w} = \frac{8}{13}$$
$$1{,}456 = 1{,}600 - 8w$$
$$8w = 144$$
$$w = 18$$

$\therefore$ 18 blue buttons must be removed.

(c) Let number of blue buttons to be added be y.

$$\frac{112}{200+y} = \frac{14}{27}$$
$$3{,}024 = 2{,}800 + 14y$$
$$14y = 224$$
$$y = 16$$

$\therefore$ 16 blue buttons must be added.

(d) $\text{P(blue)} = \frac{88+x}{200} = \frac{1}{2}$ or $\text{P(green)} = \frac{112-x}{200} = \frac{1}{2}$

$88 + x = 100$ $\qquad$ $112 - x = 100$

$x = 12$ $\qquad$ $x = 12$

33. Jeffrey bought a grey (G), a red (R), a blue **(B),** and a yellow (Y) T-shirt. He also bought a blue **(B)**, a white (W), and a grey (G) pair of jeans. Suppose that Jeffrey randomly matches a shirt with a pair of jeans.

(a) List all the possible ways of matching a shirt with a pair of jeans.

(b) Find the probability of Jeffrey wearing
  (i) a yellow T-shirt,
  (ii) a white pair of jeans.

(c) Let M be the event that Jeffrey matches a shirt with a pair of jeans of the same color. Find P(M) and P(M').

Solution

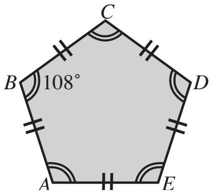
(a) Sample space, S = { (G, B), (G, W), (G, G), (R, B), (R, W), (R, G), (B, B), (B, W), (B, G), (Y, B), (Y, W), (Y, G)}.

(b) **(i)** $\text{P(yellow T-shirt)} = \frac{3}{12} = \frac{1}{4}$

(ii) $\text{P(white pair of jeans)} = \frac{4}{12} = \frac{1}{3}$

(c) $P(M) = P[(B, B), (G, G)]$
$$= \frac{2}{12}$$
$$= \frac{1}{6}$$
$$P(M') = 1 - \frac{1}{6}$$
$$= \frac{5}{6}$$

34. $ABCDE$ is a 5-sided plane type with sides of equal lengths. $m\angle ABC = m\angle BCD = m\angle CDE = m\angle DEA = m\angle EAB = 108°$.
A triangle is drawn at random using three of the points A, B, C, D, and E as vertices.



(a) List the sample space.

(b) Let X be the event that A is a vertex of the drawn triangle. Find P(X) and P(X').

(c) Let Y be the event that all the angles of the drawn triangle are acute angles.
  (i) Express Y using the listing method.
  (ii) Find P(Y).

Solution

(a) Sample space, S = {ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE}.

(b) $P(X) = \frac{6}{10}$
$$= \frac{3}{5}$$
$$P(X') = 1 - \frac{3}{5}$$
$$= \frac{2}{5}$$

(c) **(i)** $m\angle ABC = m\angle BCD = m\angle CDE = m\angle DEA = m\angle EAB$
$$= \frac{(5-2)\times 180°}{5}$$
$$= 108°$$
$\therefore \angle ABC, \angle BCD, \angle CDE, \angle DEA,$ and $\angle EAB$ are obtuse angles.
Y = {ACD, BDE, ACE, ABD, BCE}

(ii) $\therefore P(Y) = \frac{5}{10}$
$$= \frac{1}{2}$$

Enrichment

35. Let A = {apple},
B = {banana, mango},
C = {cherry, mango, pear}.

(a) List all the possible subsets of
  (i) A,
  (ii) B,
  (iii) C.

(b) If a set P has n elements, state the number of possible subsets of P.

(c) Suggest a universal set ξ for the sets A, B, and C.

Solution

(a) **(i)** The possible subsets of A are:
ϕ, {apple}.

(ii) The possible subsets of B are:
ϕ, {banana}, {mango}, {banana, mango}.

(iii) The possible subsets of C are:
ϕ, {cherry}, {mango}, {pear}, {cherry, mango}, {cherry, pear}, {mango, pear}, {cherry, mango, pear}.

(b) The number of possible subsets of $P = 2^n$.

(c) ξ is the set of all fruits.

36. A telemarketing salesperson selects telephone numbers randomly from a telephone directory.

(a) If one number is selected, what is the probability that the last two digits of the number are the same?

(b) If 3 numbers are selected, what is the probability that the last digits of the 3 numbers are the same?

(c) If 11 numbers are selected, what is the probability that at least two numbers have the same last digit?