Chapter 1
Setting the Foundation

1.1 God and the Laws of Math

Math — particularly upper-level math like algebra — has become associated in many people's minds with confusing rules and endless equations that seem disconnected from reality and pointless to learn.

$$TC(Q, q_i, m_i) = \sum_{i=1}^{n} \left[\frac{D_i}{m_i q_i} S_i + c_i^{v} D_i + \frac{q_i H_i^{v}}{2} \left(m_i \left(1 - \frac{D_i}{P_i}\right) - 1 + 2\frac{D_i}{P_i} \right) \right]$$

But math — including algebra — is *not* disconnected from reality . . . and it's not pointless to learn! Regardless of your past experience in math, we invite you to join us on an exciting journey. Yes, we will have to grapple with rules and equations, but we'll do so while seeing how they apply outside of a textbook and how they proclaim the praises of the Creator.

Yes, you heard that right: Math declares praise to God. We live in a consistent universe — a universe so consistent, in fact, that we can record those consistencies and rely on them to hold true day after day, year after year. And that is exactly what we're doing in math! Every time you jump up, you come back down to the ground (we call that the law of gravity). Every morning the sun rises and in the evening it sets (the earth rotates around its axis once every 24 hours). Every time you plug a power cord into a wall outlet, you power its device (the laws of electromagnetism). All of these consistencies can be described using math.

For example, we can use letters to represent the consistent relationship between the force applied to an object (F), the velocity (v — think speed in a certain direction) of that object, and the power (P) produced: $P = Fv$. This relationship holds true over and over and over again because this universe operates in a predictable way, making modern science possible.

This relationship assumes the force is in the same direction as the velocity.

$$P = Fv$$

$$Power = Force\ (velocity)$$

Now why is the universe so consistent? Why do mathematical laws hold true? Because that's how God set up the universe, and day after day, year after year, He is continuing to keep His "covenant" with the "fixed order" around us — an order that math helps us describe.

> *Thus says the LORD: If I have not established my covenant with day and night and the fixed order of heaven and earth, then I will reject the offspring of Jacob and David my servant and will not choose one of his offspring to rule over the offspring of Abraham, Isaac, and Jacob. For I will restore their fortunes and will have mercy on them. (Jeremiah 33:25–26; ESV)*

Do you catch what God is saying here in Jeremiah? He's telling His people to look around and see how faithful He is to sustain the consistencies around them and telling them He'll be just as faithful to keep His promise to them too. The very consistencies around us — which math helps us record — serve as a testimony to what a faithful, covenant-keeping God we have! Math should continually remind us that we *can* trust God.

Are you beginning to catch a glimpse of how exciting math can be? Every problem you solve and new consistency you learn about is shouting out at you that God is still faithful, keeping His covenant with the "fixed order" . . . and it's that same faithful God who has promised to save all who call upon Him (Romans 10:13) and to complete the work He begins (Philippians 1:6).

Math is ultimately a way of describing the "fixed order" God put in place and sustains (by the word of His power, no less! (Hebrews 1:3)).

For years now, you've been adding, subtracting, multiplying, and dividing numbers. All of these processes are known as mathematical **operations**. When we add, we're really describing how God determined quantities to combine. When we multiply, we're really describing how God causes sets of quantities to combine (4 *times* 3 means 4 *sets of* 3). All of math is a way of describing the "fixed order" God made all around us.

You've also likely learned about various **properties** in math. Properties, like operations, are ways of describing the "fixed order" God created and sustains. The key properties of arithmetic are below as a review — make sure you know them, as we will continue to build on them throughout this book.

Note that because of the consistent way God holds all things together, we can rely on these properties to hold true for all numbers **and generalize the relationships using letters to stand for any number**.

Properties

Commutative Property of Addition and Multiplication

Order doesn't matter.

Addition	Multiplication
$a + b = b + a$	$ab = ba$

The *a* and *b* could stand for any number — we're just saying it doesn't matter which number comes first, the answers will be the same. Note that ab means a times b — when we're using letters to stand for numbers, we don't have to bother to write out a multiplication sign; just putting them next to each other means to multiply them.

Below is an illustration with actual numbers plugged in for the a and b placeholders.

Addition	Multiplication
$1 + 2 = 2 + 1$	$2(3) = 3(2)$
$3 = 3$	$6 = 6$

Note that the **parentheses** in 2(3) means to multiply. A number or symbol next to a parenthesis means to multiply by whatever is inside the parentheses.

Associative Property of Addition and Multiplication

Grouping doesn't matter. The grouping would be useful if there were different operations and we wanted the problem solved in an unusual order — but they don't affect anything when it's all addition or all multiplication, as we'll get the same answer no matter how we group numbers being added or multiplied.

Addition	Multiplication
$(a + \mathrm{b}) + c = a + (b + c)$	$(ab)c = a(bc)$

Again, here's an example with numbers plugged in to the placeholders:

Addition	Multiplication
$(1 + 2) + 3 = 1 + (2 + 3)$	$(2 \cdot 3)4 = 2(3 \cdot 4)$
$3 \quad + 3 = 1 + \quad 5$	$(6)4 = 2(12)$
$6 = 6$	$24 = 24$

Notice that we used several different ways of showing multiplication in this box — remember, ×, •, quantities right next to parentheses, and letters written next to each other all mean multiplication.

Identity Property of Addition and Multiplication	
Adding 0 doesn't change the value.	*Multiplying by 1 doesn't change the value.*
Addition	**Multiplication**
$a + 0 = a$	$1a = a$
Example: $2 + 0 = 2$	*Example:* $1(2) = 2$

Properties of Division

Any Number (Except 0) Divided by Itself Equals 1.

$$a \div a = 1, \text{ provided } a \neq 0$$

Division by Zero

Strictly speaking, we can't divide by 0, as you can't divide something by nothing. So, if you encounter something like 10 divided by $a - 2$ (which would be written $\frac{10}{a-2}$), note that a cannot equal 2, or else we would be dividing by 0, since 2 – 2 equals 0.

In higher level math such as calculus, you will learn there are more nuanced ways of looking at these seemingly impossible divisions.

Math Is a Useful Tool

Because math helps us describe the "fixed order" God put in place around us, it's useful outside of a textbook! You may have noticed the word "applied" in the subtitle of this course. We want you to be equipped to use the concepts you learn outside of a textbook — to apply them.

Walter W. Sawyer compares mathematics to "a chest of tools" and urges students to know how to use them.

> *Mathematics is like a chest of tools: Before studying the tools in detail, a good workman should know the object of each, when it is used, how it is used, what it is used for.*[1]

Tools come in all sorts of layers of complexity. There are very simple tools, such as a hammer. And then there are tools like high-powered routers that serve a specific task. These latter tools take longer to learn how to use and serve a more specialized function, but they're very powerful for what they do.

Some of the concepts we'll be studying in this course are like that high-powered router. They are incredibly useful, but are more focused in their application. Thus you may not find yourself using all of them on a regular basis . . . or even at all, depending on the field you pursue. However, by learning how to use these high-powered tools, you'll both better understand the complexities of God's creation (and how the myriad of technological devices around us work) and be better equipped to think through other types of problems you might encounter. In other words, they can help you both grow in appreciation for the Creator and learn problem-solving and critical-thinking skills that can then be transferred to other areas of life and can help you complete the tasks God gives you to do. Above all, studying these tools will give you a deeper look at how math does, indeed, describe God's creation and proclaim God's praises.

Introducing Algebra and Formulas

This course is specifically an **algebra** course, meaning we're going to focus on the branch of math in which "letters and other general symbols are used to represent numbers and quantities in formulae and equations."[2] So you're going to see a lot of letters standing for quantities.

While we'll soon move on to not only using letters to stand for unknown quantities, then also determining the value of those "unknown quantities by means of those that are known,"[3] but we're also going to start in these first several chapters by reviewing the basics.

One basic application of algebra you've been using for years is that of allowing us to represent formulas. A **formula** is simply "a mathematical relationship or rule expressed in symbols."[4] You've been using formulas for years. For example, to find the circumference of a circle, you multiply the irrational number π, that begins 3.14, by the diameter.

$$Circumference = \pi(diameter)$$

Rather than writing all of those words out, we typically express this relationship as a formula. **Notice that we've used letters to stand for the circumference and distance!**

$$C = \pi d$$

To use the formula, we replace the letters with the values for that particular circle. For example, if we know a circle has a diameter of 4 in, we would have this:

$$C = \pi(4 \text{ in}) \approx 3.14(4 \text{ in}) \approx 12.56 \text{ in}$$

Note: We used an approximate value of 3.14 for π. When solving problems in this course, you can either use whatever rounded value you've memorized, or simply press the π button on your calculator to use the rounded value it has stored.

To use a formula, insert the appropriate values for the various letters and simplify!

Notice that we used an **approximately equal sign** ($\approx$). We did that because we used a rounded value for π — thus the answer is not an exact answer. In this course, we'll use the approximately equal sign whenever we round just to make it clear we're using rounded values.

As another example, suppose the diameter of a circular hot tub was 5.62 ft. In that case, we would find the circumference like this:

$$C = \pi(5.62 \text{ ft}) \approx 3.14(5.62 \text{ ft}) \approx 17.6468 \text{ ft} \approx 17.647 \text{ ft}$$

Since this is an applied algebra course, you may find yourself working with units of measure more than in your previous math courses. We'll walk through how to work with them as we go.

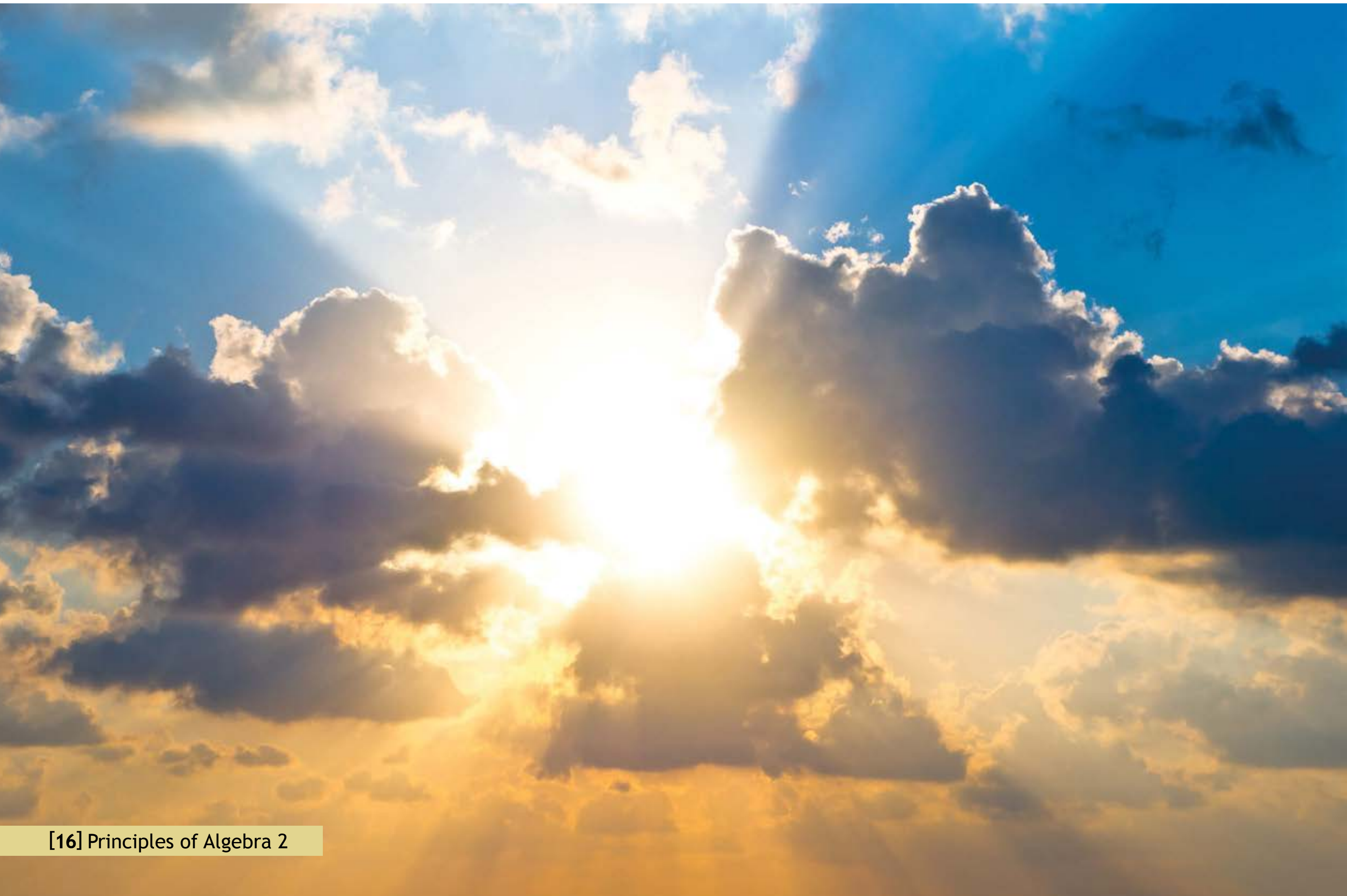
Notice that we rounded to the 3rd decimal place. In this course, **always round to the 3rd decimal place unless instructed otherwise . . . and use the ≈ to show that you did!** (If needed, see the endnote for a reminder on how to round.[5])

Also notice that we included the unit of measure in the answer. **Always include units of measure in the answer if one is given.** Why? Because otherwise no one knows what the answer really is. 17.647 could be 17.647 centimeters, pounds, $ ... we need units to know what is being represented!

Keeping Perspective

As we embark on an exploration of math in this course, always remember that math is simply a way of describing the "fixed order" God created and sustains.

Above all, remember that math is shouting out at you that just as faithful as God is to keep His covenant with the "fixed order" of creation, He will be just as faithful to everything else He's said in His Word, the Bible. This should give you incredible peace if you've placed your trust in His way of salvation — Jesus — as God then promises to save and sanctify you. But it should give you great fear if you're trusting in anything else for salvation, as God says that no one comes to God apart from Jesus (John 14:6).[6] Make sure your trust is in Jesus (see Appendix A: Math's Message). Then get ready for an exciting journey into exploring His "fixed order."



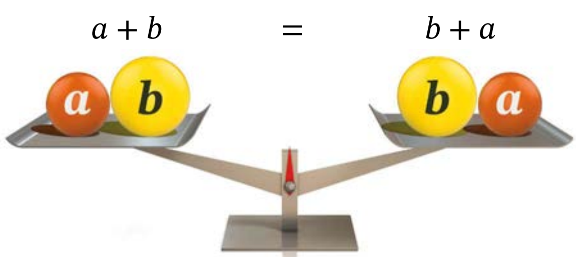
1.2 The Language of Mathematics

If we're going to describe the "fixed order" around us, we need a language to use! And just like learning any spoken language takes work, so too does it take work to learn the language of mathematics. But once you know the language, it makes communication simple. We can think of the language side of mathematics as a **convention**, or "a way in which something is usually done, especially within a particular area or activity."[7]

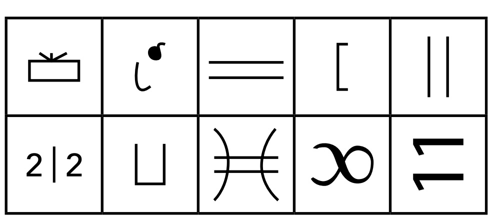
Just as words help us describe quantities and concepts, so can symbols. In math, though, we prefer symbols, since it's a lot easier to work with them. (Imagine trying to add a million, four hundred thirteen thousand, seven hundred nineteen with four thousand, three hundred ninety without first rewriting it using symbols as 1,413,719 + 4,390.)

We have symbols to stand for quantities with a known value, such as the symbol 1, 0.56, π, etc. We also have symbols to stand as placeholders for unknown quantities, such as a, v, V, σ, etc. Notice that when we use a letter, **the case matters**. V is representing a different value than v. (In fact, you might encounter them both in the same problem, with V standing for volume and v for velocity.)

You've already learned a lot of symbols — the plus and minus signs, different ways to multiply (see box), etc. You know that an **equal sign** (=) is an agreed upon way to represent that the quantities on both sides of the sign have the same exact value.



This symbol, though, is really just a convention for representing the consistency God created and sustains. Note that we could have used different symbols to show equality — in fact, below are a few that have historically been used.[8] The symbol is simply part of the language system we're using to communicate about the consistencies around us.



Sometimes we want to compare quantities that are not equal. In that case, we're working with an **inequality**. And you've already learned various symbols to help you describe inequalities.

Hint: When using the less than and greater than signs ($<$ and $>$), put the smaller side of the sign with the smaller quantity.

6 does not equal 7;
$6 \neq 7$

6 is less than 7;
$6 < 7$

7 does not equal 6;
$7 \neq 6$

7 is greater than 6;
$7 > 6$

Different Symbols for Showing Multiplication

Using a $\times$ or • sign or putting quantities right next to parentheses are all ways of showing multiplication. The expressions below all mean 5 *times* 2.

$$5 \bullet 2 \qquad 5 \times 2 \qquad 5(2)$$

Note: Since the letter x is used often for unknowns in algebra and is hard to distinguish from $\times$ when handwritten, be careful when using $\times$ to mean multiplication. Parentheses is often a better choice in algebra.

If we're using letters to stand for quantities, we don't need to include a multiplication sign (***ab* means *a* times *b***, and $5b$ means 5 times b). This is an agreed-upon convention to simplify expressing multiplication.

When exploring the language side of mathematics, it's important to keep in mind that symbols and other conventions can (and sometimes do!) vary. Just like language systems have varied since the Tower of Babel, the way we express mathematical concepts also varies (Genesis 11). Much of math consists of conventions: agreed-upon protocols or rules that aid us in communication. Man is only able to develop conventions because God made us in His image, capable of subduing the earth (Genesis 1:27).

Symbols for Units of Measure/Conventions in Showing Units

As we apply algebra, we'll encounter numbers that have units of measure quite frequently. After all, if we're weighing something, we need to know if that weight is in pounds, newtons, or some other unit.

There are many different conventions regarding how to write units of measure. Appendix B: Reference Section lists many of the units of measure you'll encounter and the abbreviations used in this course. It's worth noting, though, that those abbreviations can (and do!) vary. For example, while we'll abbreviate seconds as s, you may see it abbreviated elsewhere as sec. The abbreviations we choose are just conventions.

It's also worth noting that **we work with units of measure much the same way as we do with letters we're using to stand for quantities**, only we need to be careful to treat the whole abbreviation as the unit. For example, we view kg as standing for kilograms, *not* as a k multiplied by a g.

> Note that in this course, we will italicize letters standing for unknowns, but **we will not italicize letters that are a unit of measure.**

Let's say we were to multiply 7 kg by 3 m. What do we do with the units of measure? We multiply them!

$(7 \text{ kg})(3 \text{ m}) = 21 \text{ kg} \cdot \text{m}$ *or* 21 kg m (pronounced "21 kilogram-meters")

Notice that with units, we didn't just put the two units next to each other (i.e., we didn't write kgm) like we would when using unknowns. It would be too easy then to think the entire unit was kgm, or that we meant k times g times m, rather than that we have a unit of kg *times* m. Some resources, though, will just put a space between the two units and leave them next to each other, like 21 kg m. Either way clearly shows the unit of measure.

Now you might be wondering what a "kilogram-meter" is. Often the units we come to in problems don't have a specific definition like a kilogram or a meter does. In this case, a kilogram-meter is simply the result of multiplying kilograms by meters. (We can use it to measure the work done by applying a force, such as to a pump.[9])

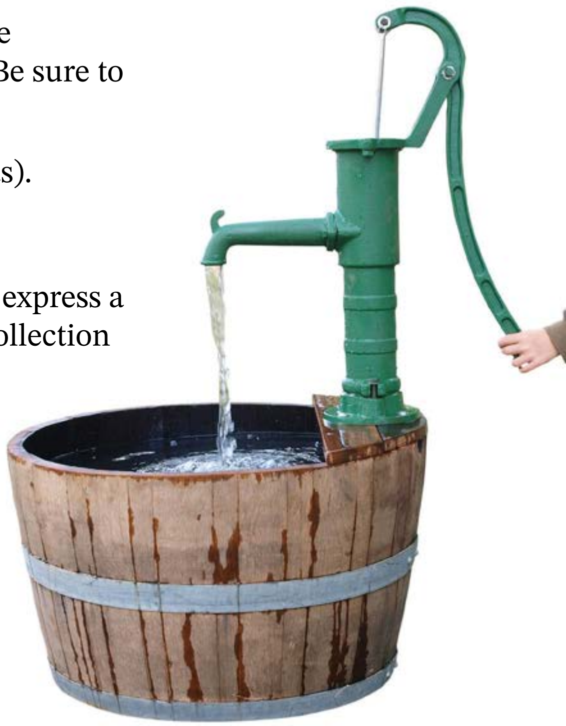
Words Used in Math

Speaking of the language side of math, here are a few words we'll use extensively as we explore the principles of applied algebra together. Be sure to familiarize yourself with them before we get started.

Constants – Quantities that have a fixed value (5 and π are constants).

Variables – Values whose value can vary in a problem (such as x).

Expression — An expression is "a collection of symbols that jointly express a quantity."[10] For example, $4 + 5$ is an expression — 4, 5, and + are a collection of symbols that together express the quantity 9. Likewise, $a + b$ is an expression. The a and b here are simply placeholders for numbers.



Equation — An equation is "a statement that the values of two mathematical expressions are equal (indicated by the sign =)."[11]

Simplify — When we refer to simplifying an expression or equation, we mean to express it as simply as possible. For example, 5 + 6 simplifies to 11.

We'll explore the definition of constants and variables further in Chapter 7.

Unless instructed otherwise, **simplify your answers as much as possible** in this course.

Keeping Perspective

On your worksheet today, you're going to review some conventions we'll be following in this course. Make sure you're familiar with them so that you can follow along as we go forward. And remember to be patient with yourself as you try to learn the language side of mathematics — it takes work to learn another language!

1.3 Understanding, Multiplying, and Dividing Fractions

Throughout this first chapter we're going to be reviewing some common foundational conventions. Most of these conventions should be familiar, but we are hoping you will see them in a new light as we both review them *and* see how they apply outside of a textbook. We're going to start with fractions.

Understanding Fractions

A lot of the confusion stems from the fact that the word "fraction" is used in different ways. A fraction can refer to

- a partial quantity, no matter how it's written. *Examples:* $\frac{1}{4}$, 0.25, 25%
- a specific notation with a **numerator** (top number) showing the number of parts and a **denominator** (bottom number) showing the parts in a whole. *Examples:* $\frac{1}{4}, \frac{8a}{c}, \frac{5+d}{5}$

Historical Tidbit

Did you know that fractions at one point were written without a line? So $\frac{4}{5}$ would have been $\begin{smallmatrix}4\\5\end{smallmatrix}$ instead. Leonardo Pisano "was one of the first to separate the numerator from the denominator by a fractional line."[12] Remember, while God created the real-life quantities and consistencies fractions describe, the notations and conventions used to describe them have (and still do!) vary. God made man in His image with creativity to develop new ways of exploring and describing His creation.

However, there's a third meaning to the word "fraction," and it's this third meaning that will help you really understand how we use this notation to describe God's creation.

- Fractions are simply a convention for representing division.

If I want to write $1 \div 4$, I can do this more easily using a fraction line: $\frac{1}{4}$.

Now you may be used to thinking of $\frac{1}{4}$ as one part out of four . . . and it is. If you take 1 sandwich and you divide it into 4 pieces, each piece is $\frac{1}{4}$ of the whole.

The $\frac{1}{4}$ here could be thought of as the division problem *or* as the number of parts over the number of parts in the whole.

Understanding that the fraction line can be thought of as a division symbol is critical for algebra, as we typically won't be using a division sign — just the fraction line. If we want to write $5a \div b$, we will simply write $\frac{5a}{b}$. Likewise, to write $9 \div 2$, we could write $\frac{9}{2}$. Notice that this notation is more concise. It proves quite useful.

> We call $\frac{9}{2}$ an **improper fraction** because its numerator (9) is greater than its denominator (2). While in other math courses you might have rewritten this as a **mixed number** (a number with a whole part and a fractional part — $4\frac{1}{2}$ in this case), in algebra **we avoid mixed numbers.** The reason is that in algebra, we work a lot with letters, which get written right next to numbers to show multiplication. Thus it'd be easy to accidentally mistake $4\frac{1}{2}$ for 4 times $\frac{1}{2}$ rather than as 4 wholes and $\frac{1}{2}$. In this course, **do not use mixed numbers.**

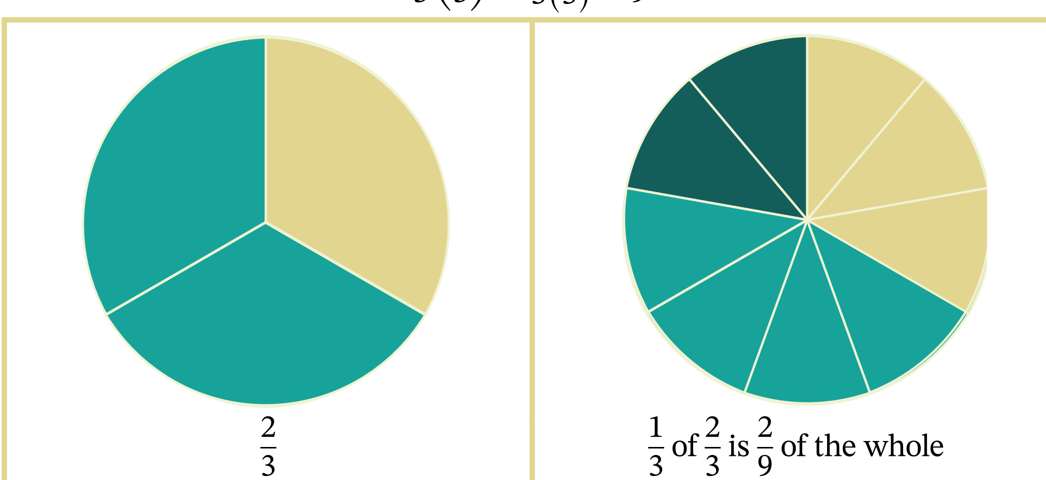
Keeping this in mind, let's briefly review how to multiply and divide fractions. It's important that you thoroughly know how to work with them, whether dealing with known quantities or with letters standing for unknown quantities.

Multiplying Fractions

We're used to thinking of multiplication in terms of repeated addition. If we want to figure out how much we'll pay if each ticket costs \$2 and we want 3 of them, we multiply 3 times \$2 to add up the cost of 3 tickets. But when we multiply by a fraction with a value *less than 1*, we're finding a quantity *of* another quantity. For example, $\frac{1}{3}\left(\frac{2}{3}\right)$ is finding a third *of* two thirds. We might need to find this if we were thirding a recipe that called for $\frac{2}{3}$ cup of flour and trying to figure out how much flour to put in instead.

How do we actually perform the multiplication? **We multiply fractions by multiplying the numerators together and the denominators together.**

$$\frac{1}{3}\left(\frac{2}{3}\right) = \frac{1(2)}{3(3)} = \frac{2}{9}$$

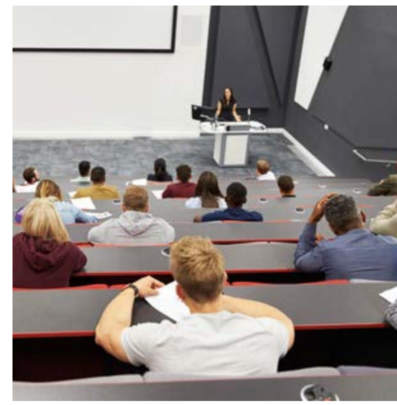


$\frac{2}{3}$ $\qquad$ $\frac{1}{3}$ of $\frac{2}{3}$ is $\frac{2}{9}$ of the whole

Notice that we can multiply the numerators and denominators when we have used a letter(s) to stand for an unknown value(s) too. Why? Because of the consistent way God governs all things!

Example: $\frac{a}{5c}\left(\frac{3}{b}\right) = \frac{a(3)}{5c(b)} = \frac{3a}{5bc}$

In the above example, note how we listed the 3 and the 5 first in our final answer, making the numerator $3a$ instead of $a3$. Also notice that we wrote $5cb$ as $5bc$. Except in some science or application situations, **it's standard to list constant values in a term first, then all unknown quantities (represented by letters), in alphabetical order.**



Sometimes, we need to multiply a fraction by a whole number. Once again, if the fraction is less than 1, we're finding a portion *of* another quantity. For example, $\frac{1}{3}(30)$ is $\frac{1}{3}$ of 30. If we want to figure out what $\frac{1}{3}$ of a class of 30 college students is, we'd multiply $\frac{1}{3}(30)$.

On the flip side, if we wanted to multiply a recipe by 30 to make enough for a crowd and the recipe calls for $\frac{1}{3}$ c of flour, we'd be finding $30\left(\frac{1}{3}\right)$ . . . which is finding repeated addition (adding $\frac{1}{3}$ thirty times).

And how do we complete the multiplication in either case? Well, since any number divided by 1 equals itself, we can think of 30 as $\frac{30}{1}$. Thus, we can multiply a fraction by whole numbers the same way we would fractions: just view the whole number or unknown as a numerator of a fraction with a 1 as the denominator.

$$\frac{1}{3}(30) = \frac{1}{3}\left(\frac{30}{1}\right) = \frac{30}{3} = 10$$

($\frac{1}{3}$ of a class of 30 students is 10 students)

$$30\left(\frac{1}{3}\right) = \frac{30}{1}\left(\frac{1}{3}\right) = \frac{30}{3} = 10$$

(taking $\frac{1}{3}$ c of flour 30 times results in 10 c of flour)

It's not necessary to write out all of the steps shown in this first example. They're just here to help clarify.

Example: $5\left(\frac{a}{b}\right) = \frac{5}{1}\left(\frac{a}{b}\right) = \frac{5a}{1b} = \frac{5a}{b}$

Example: $4ac\left(\frac{b}{d}\right) = \frac{4abc}{d}$

Notice that we listed the letters in alphabetical order. $4acb$ and $4abc$ mean the same thing, as multiplication is commutative and associative.

Example: $x\left(\frac{y}{z}\right) = \frac{x}{1}\left(\frac{y}{z}\right) = \frac{xy}{z}$

Dividing Fractions

To divide by a fraction, we invert (i.e., take the multiplicative inverse of) and multiply. This "rule" is a shortcut to help us quickly find an answer. There is another way to do it . . . only this is simpler. And again, we can do it with either known or unknown quantities.

$$\frac{1}{2} \div \frac{2}{3} = \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}$$

inverted and multiplied

$$\frac{x}{b} \div \frac{2a}{3} = \frac{\frac{x}{b}}{\frac{2a}{3}} = \frac{x}{b}\left(\frac{3}{2a}\right) = \frac{3x}{2ab}$$

inverted and multiplied

Let's think through for a minute what we're really finding with division. Division is "the action of separating something into parts or the process of being separated."[13] This holds true for fractions as well, only it's important to realize that when we divide by a fraction with a value less than 1, we end up with a *greater value.* For example, $30 \div \frac{1}{4} = 30\left(\frac{4}{1}\right) = 120$. One example of this is if we have \$30 and divide it up into quarters where each quarter is $\$\frac{1}{4}$ (i.e., 25 cents = \$0.25 = $\$\frac{1}{4}$), we'd get 120 quarters.

Or consider the example we looked at under multiplication with thirding a recipe. If we need to third a recipe calling for $\frac{2}{3}$ c flour, we can either look at this as finding $\frac{1}{3}$ of $\frac{2}{3}$, or $\frac{1}{3}\left(\frac{2}{3}\right) = \frac{2}{9}$, or as dividing $\frac{2}{3}$ by 3. $\frac{2}{3} \div 3 = \frac{2}{3}\left(\frac{1}{3}\right) = \frac{2}{9}$. We get the same answer either way. And notice that, as with multiplication, **we viewed the whole number (the 3) as $\frac{3}{1}$ and then inverted and multiplied.**

We explored the "rules" of working with fractions — such as inverting and multiplying — much more in *Principles of Mathematics, Book 1.* We're just reviewing them briefly here, but please see the earlier book to see how each rule really does rest on the consistent way God causes objects to operate. Also note that if the numerator and the denominator fully divide by the value we're dividing by as in $\frac{32}{25} \div \frac{8}{5}$, we can simply divide the numerators and the denominators: $\frac{32 \div 8}{25 \div 5} = \frac{4}{5}$. The "invert and multiply" rule comes in handy, though, when the numerators or the denominators don't divide evenly, as in $\frac{1}{5} \div \frac{2}{3}$. Were we to simply divide the numerators and denominators here, we'd end up with $\frac{1 \div 2}{5 \div 3}$, *or* $\frac{\frac{1}{2}}{\frac{5}{3}}$, which isn't very simplified.

Multiplicative Inverse Review

Multiplicative inverse is a name to describe the number that, when multiplied by another number, equals 1. Some people also call it the **reciprocal** of a number, or, so long as the context is clear, simply the inverse. **In a fraction, the denominator becomes the numerator and the numerator the denominator.**

$\frac{7}{2}$ is the multiplicative inverse of $\frac{2}{7}$, as $\frac{7}{2}\left(\frac{2}{7}\right) = \frac{7(2)}{2(7)} = \frac{14}{14} = 1$

$\frac{b}{3}$ is the multiplicative inverse of $\frac{3}{b}$, as $\frac{b}{3}\left(\frac{3}{b}\right) = \frac{3b}{3b} = 1$

(As we saw in Lesson 1.1, any quantity except 0 divided by itself equals 1. Note that here we're assuming b does not equal 0; $\frac{3b}{3b}$ can't be simplified if $b = 0$.)

Remember, the fraction line means division.

So $\frac{\frac{4}{5}}{\frac{2}{7}}$ means $\frac{4}{5}$ divided by $\frac{2}{7}$.

Example: $\frac{\frac{1}{5}}{\frac{2}{7}} = \frac{1}{5} \div \frac{2}{7} = \frac{1}{5}\left(\frac{7}{2}\right) = \frac{1(7)}{5(2)} = \frac{7}{10}$

Example: $\frac{\frac{a}{5}}{\frac{3}{b}} = \frac{a}{5}\left(\frac{b}{3}\right) = \frac{ab}{5(3)} = \frac{ab}{15}$

To keep things simple to grade, if a problem is given to you in fractional form and there's a fractional part in the answer, give your answer as a fraction. If there's a decimal in the problem and there's a fractional part in the answer, then give the answer using a decimal. If a problem has both fractions and decimals, you can give your answer in either form.

Keeping Perspective

As you review fractions, remember that we work with them because they help us describe God's creation. And the various methods and "rules" help us accurately describe the real-life principles God created and holds together.

1.4 Equivalent Fractions and Simplifying Fractions

It's time now to continue reviewing fractions by exploring the concept of equivalent fractions and simplifying fractions. Both of these concepts should already be familiar to you, but they're so important that they are worth a review. Not only does simplifying fractions help us better represent information in an easier-to-process way (it is easier to instantly know what quantity is meant by $\frac{1}{3}$ than by $\frac{30}{90}$), but it also helps us add and subtract fractions . . . and even find unknown quantities! In short, it's a skill we'll be using a lot.

Equivalent Fractions

Even though we don't know a's value, we know that we have some value divided by that same value. Because of the consistent way God governs all things, we know the answer will be 1 (as long as $a \neq 0$).

Recall from Lesson 1.1 that multiplying by 1 doesn't change the value (the identity property of multiplication). Since any number divided by itself equals 1, then $\frac{8}{8}$ or $\frac{a}{a}$ are fractions worth 1 (remember, $\frac{8}{8}$ means 8 divided by 8, and $\frac{a}{a}$ means a divided by a).

It follows **that if we multiply a fraction by a fraction equal to 1, we're not changing the value.** Instead, we're forming what we call an **equivalent fraction**.

Example: $\frac{4}{2}\left(\frac{8}{8}\right) = \frac{4(8)}{2(8)} = \frac{32}{16} = 2$

Notice that multiplying by $\frac{8}{8}$ didn't change the value. $\frac{4}{2}$ (which means 4 *divided by* 2) equals 2, as does $\frac{32}{16}$ (which means 32 *divided by* 16).

Because of the consistent way God governs all things, this holds true for unknown quantities as well.

Note that we wrote the numerator as ac instead of ca and $8ab$ instead of $8ba$. Since multiplication is commutative and associative, the order and grouping doesn't matter. We put the a first, though, as it's a convention to list the letters in alphabetical order. Notice how it is easier to read $\frac{ac}{8ab}$ than in $\frac{ca}{8ba}$.

Example: $\frac{c}{8b}\left(\frac{a}{a}\right) = \frac{ac}{8ab}$

Knowing this can help us both simplify and add and subtract fractions, as we'll soon review.

Whole Numbers and Equivalent Fractions

We can form equivalent fractions for whole quantities too!

Example: $3\left(\frac{2}{2}\right) = \frac{3(2)}{2} = \frac{6}{2}$

Notice that $\frac{6}{2}$, which represents 6 *divided by* 2, does indeed equal 3.

Notice also that we viewed the 3 as a numerator. Since any number divided by 1 equals itself, we could have rewritten 3 as $\frac{3}{1}$ to clarify this.

$$3\left(\frac{2}{2}\right) = \frac{3}{1}\left(\frac{2}{2}\right) = \frac{3(2)}{2} = \frac{6}{2}$$

The same thing applies for unknowns as well.

Example: $a\left(\frac{x}{x}\right) = \frac{a}{1}\left(\frac{x}{x}\right) = \frac{ax}{x}$

Simplifying Fractions

We often need to simplify fractions. A simple way to think about this is to think about completing some of the division ahead of time.

Remember, the fraction line means to divide. So, if we can complete part of that division, that helps simplify the fraction.

You've already been doing this for years. For example, when you've seen problems like $\frac{18}{4}$, you see that both the numerator and the denominator can be divided by 2. So, you go ahead and complete that part of the division, simplifying the fraction down to $\frac{9}{2}$.

We could also think of this as dividing both the numerator *and* the denominator by 2.

$$\frac{18 \div 2}{4 \div 2} = \frac{9}{2}$$

Another way of thinking about it is that we're looking at the factors that make up the number and then seeing what is repeated in the numerator and denominator.

$$\frac{18}{4} = \frac{9 \bullet 2}{2 \bullet 2}$$

Notice we can then see that there's a 2 in both the numerator and the denominator. This is really a fraction worth 1!

$$\frac{18}{4} = \frac{9 \bullet 2}{2 \bullet 2} = \frac{9}{2}\left(\frac{2}{2}\right)$$

And since multiplying by 1 doesn't change the value, it follows that we can simply remove the $\frac{2}{2}$.

$$\frac{18}{4} = \frac{9 \bullet 2}{2 \bullet 2} = \frac{9}{2}\left(\frac{\cancel{2}}{\cancel{2}}\right) = \frac{9}{2}$$

$\frac{18}{4}$ and $\frac{9}{2}$ represent the same amount. Notice that we didn't have to rewrite the whole fraction; we could have just crossed out the 2 in the numerator and denominator.

$$\frac{18}{4} = \frac{9 \bullet \cancel{2}}{2 \bullet \cancel{2}} = \frac{9}{2}$$

Simplifying fractions becomes super important when we're dealing with unknowns, as it helps us solve problems we couldn't otherwise. For example, suppose we had $\frac{ac}{8a}$ and we knew the value of c but did *not* know the value for a. What can we do? Well, we know that, provided a is not 0, $\frac{a}{a}$ equals 1, as, due to the consistent way God governs all things, any number except 0 divided by itself equals 1. So, we can go ahead and complete that part of the division by crossing the a's out.

> Unless told otherwise, you can assume when simplifying fractions in this course that the expression in the denominator does not equal 0.

$$\frac{ac}{8a} = \frac{\cancel{a}c}{8\cancel{a}} = \frac{c}{8}$$

Another way to think about what we just did is dividing both the numerator and the denominator by the *same amount.*

$$\frac{ac \div a}{8a \div a} = \frac{c}{8}$$

We can also think of it as removing the fraction worth one, since we know that won't affect the value.

$$\frac{ac}{8a} = \frac{c}{8}\left(\frac{a}{a}\right) = \frac{c}{8}$$

> Be careful! You can only complete some of the division in a fraction when you're dealing with multiplication. If you had $\frac{a + c}{8a}$ instead of $\frac{ac}{8a}$, you couldn't simply cross out the a's, as you're being asked to add a and c and *then* divide that amount by $8a$. It's only when you can view the numerator and denominator as a product of factors (i.e., you're dealing with multiplication!) that you can cancel out.

Keeping Perspective

Notice how in this lesson we built on the identity property of multiplication to help us form equivalent fractions and simplify them. As we dig deeper into math, we will be continuing to build. But always remember that everything rests on those consistencies God created and sustains!

Unless instructed otherwise, always simplify fractions in this course as much as possible. $\frac{10}{4x}$ should be simplified to $\frac{5}{2x}$ in your answer. (We divided both the numerator and the denominator by 2.) Simplifying makes it easier to see what's really going on at a glance and, as we mentioned at the beginning, will prove invaluable as we go forward. You'll get a chance on your worksheet to see some sample meanings for simplifying fractions, including seeing what portion of a city's population strongly supports a candidate.

1.5 Understanding Ratios and Proportions

One common application of fractions is to show ratios and proportions. As we explore these, we're going to get a chance to apply all we've looked at so far regarding fractions!

Ratios

A **ratio** is "the relative size of two quantities expressed as the quotient of one divided by the other."[14] In other words, "ratio" is a fancy name for using division to compare quantities! And since fractions represent division, they make a convenient notation to use to represent ratios.

If there are 25 students in a college class *per* only 5 computers, then the ratio between students and computers can be expressed as a fraction like this: $\frac{25 \text{ students}}{5 \text{ computers}}$. Note that we can simplify this fraction by completing the division, giving us $\frac{5 \text{ students}}{1 \text{ computer}}$, which we would read as 5 students *per* 1 computer.

There are other notations that can be used to express ratios. For example, we can write 25:5 instead of $\frac{25 \text{ students}}{5 \text{ computers}}$. But we'll use a fraction in this course.

> Typically, we don't bother to write out a 1 before a unit of measure (such as computers). We would write $\frac{5 \text{ students}}{\text{computer}}$ and read it as 5 students per computer.

If you take 2 vitamins *per* day, we could write a ratio between the number of vitamins like this: $\frac{2 \text{ vitamins}}{\text{day}}$ or $\frac{2 \text{ vitamins}}{1 \text{ day}}$.

Or say you're making a recipe, and you know that you need 2 tsp of cumin *per* 4 c of water. We could write the ratio between the cumin and water like this: $\frac{2 \text{ tsp}}{4 \text{ c}}$.

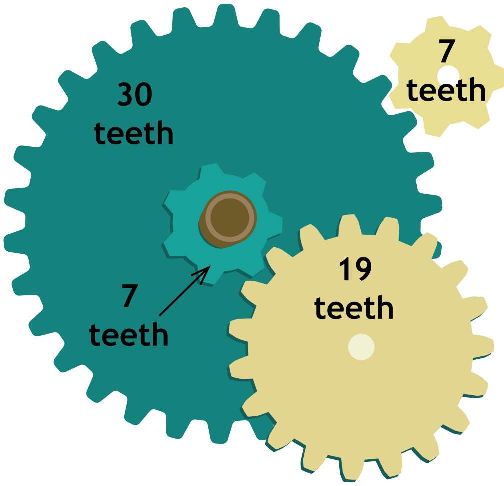
Notice that we kept italicizing *per* up above. **If you can word a problem using the word *per*, it's a good indicator that you're probably dealing with a ratio!** Remember, a ratio is simply a comparison via division . . . which you can use a fraction to represent.

> When written as a fraction, you can work with ratios just like you would with fractions!

Applying Ratios (and Multiplying Fractions, too!) Gear Ratios

Let's look at an application of ratios. Along the way, we'll also apply multiplying fractions! Consider the gears in the engine shown on the next page.

Notice how the small gear on the right has 7 teeth, or spikes, while the large gear in the middle has 30. The ratio between them is $\frac{7}{30}$, or 7 *per* 30. Let's say



that we wanted to know what portion of the large gear rotated each time the small one rotated. Notice the use of the word *of*. We need to use multiplication! We can find that by multiplying the rotation of the small gear by the rotation of the large gear. The small gear is making a complete rotation, so it's going around 1 time. The large gear, though, is only going to make it through 7 out of its 30 teeth, as that small gear only has 7 teeth (and it is the small gear's teeth rotating that cause the large gear's teeth to rotate). Thus, we'd have this:

$$1\left(\frac{7}{30}\right) = \frac{7}{30}$$

Each time the small gear goes around 1 time, the large gear is going to go $\frac{7}{30}$th of the way around, as the small gear will push it through 7 out of its 30 teeth.

Now the middle gear that only has 7 teeth is attached to the large gear via a rod rather than being moved by its teeth. Each time the large gear goes around once, it goes around once as well. That also means that each time the initial small gear goes around once, the middle gear with 7 teeth only goes around $\frac{7}{30}$th of the way around, as it's moving via a rod attached to the large gear. The final gear has 19 teeth. So each time the middle gear with 7 teeth goes around 1 time, the gear with 19 teeth will go around $\frac{7}{19}$th of the way, as the gear with 7 teeth will turn it through 7 of its 19 teeth.

$$1\left(\frac{7}{19}\right) = \frac{7}{19}$$

Keeping all this in mind, what portion *of* the 19-tooth gear rotates each time that initial 7-tooth gear rotates? Notice that we're trying to find the portion *of* something. The word *of* is a good indicator we need to use multiplication. And we do! We just say that every time the initial 7-tooth gear rotates, the middle 7-tooth gear goes $\frac{7}{30}$ of the way around, as it's attached via a rod to the large gear . . . and that each time it rotates, the final 19-tooth gear goes $\frac{7}{19}$ of the way around. So now we need to find $\frac{7}{30}$ *of* $\frac{7}{19}$, or $\frac{7}{30}\left(\frac{7}{19}\right)$.

$$\frac{7}{30}\left(\frac{7}{19}\right) = \frac{49}{570}$$

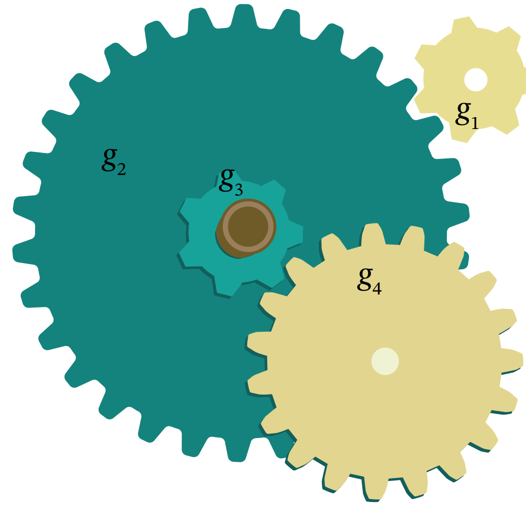
Now we've found what is called the gear ratio of the whole system: a comparison via division that, in this case, shows us the portion of the final gear that rotates each time the initial gear makes one rotation.

And since God governs a consistent universe, we could use letters to stand for the various gears and show how to find the combined gear ratio (represented by an *R*) for any size gear in this arrangement.[15]

$$R = \frac{g_1}{g_2}\left(\frac{g_3}{g_4}\right)$$

This in turn could be simplified to this:

$$R = \frac{g_1 g_3}{g_2 g_4}$$



Now we have a formula we can use to help us find the gear ratio for any gear!

Our point here is not to fully understand gears (so don't worry if you didn't follow all the details), but rather to show an application of what we reviewed today. Knowing the gear ratio helps in designing engines . . . and fractions, multiplying fractions, and ratios help us in the process!

Conversion Ratios

One common application of ratios is in converting from one unit to another. If something takes 1,800 seconds, what portion of an hour is that? We can find that like this:

$$1{,}800\ \cancel{\text{s}}\left(\frac{1\ \cancel{\text{min}}}{60\ \cancel{\text{s}}}\right)\left(\frac{1\ \text{hr}}{60\ \cancel{\text{min}}}\right) = 0.5\ \text{hr}$$

Notice that min does not mean m times i times n. It is an abbreviation standing for "minute" and is treated as a single unit.

What did we do? Well, we multiplied by $\frac{1\ \text{min}}{60\ \text{s}}$. This is really a fraction worth 1, as both 1 min and 60 s represent the *same time*. When we convert units, we multiply by fractions worth 1 so as to not change the value (we're multiplying both the numerator and the denominator by the *same* time, distance, weight, etc., only expressed in different units). These are known as **conversion ratios**.

Notice that the units canceled out when we had one in the numerator and one in the denominator like unknowns did in the last lesson. **We can work with units of measure the same way we do with unknowns**, multiplying them, dividing them, etc. So when we multiplied to convert 1,800 s, we crossed out units that were the same in both the numerator and the denominator, as they would cancel each other out.

Proportions

Connected with the idea of a ratio is that of a **proportion**, which is a fancy name for 2 equal ratios. In other words, there are 2 equivalent fractions, each of which represents a ratio.

For example, a friend who used to work at a beverage company once told us that soda companies have carefully guarded recipes for how to make their beverages . . . and about how he had to use algebra extensively to scale the recipes. Say one recipe calls for $\frac{1}{4}$ cup lemon juice *per* 1 gallon of water. Notice that this could be written as a ratio: $\frac{\frac{1}{4}\ \text{c}}{1\ \text{gal}}$. If they want to make a batch with

30 gallons of water instead, how many cups of lemon juice will they need? We'd have this proportion, where x represents the cups of lemon juice needed.

$$\frac{\frac{1}{4}\text{ c}}{1\text{ gal}} = \frac{x}{30\text{ gal}}$$

We can figure out the cups of lemon juice needed by figuring out what value would be needed to make these ratios equivalent. Notice that 30 gallons is 30 times greater than 1 gallon, as $30 \div 1 = 30$. So x must also be a value that is 30 times greater than $\frac{1}{4}$ c! Let's multiply both the numerator and the denominator of $\frac{\frac{1}{4}\text{ c}}{1\text{ gal}}$ to form an equivalent fraction with 30 gallons in the denominator.

$$\frac{\frac{1}{4}\text{c}}{1\text{ gal}}\left(\frac{30}{30}\right) = \frac{\frac{30}{4}\text{c}}{30\text{ gal}} = \frac{\frac{15}{2}\text{c}}{30\text{ gal}}$$

We would need $\frac{15}{2}$ cups.

Again, notice the use of the word *per*. Remember that if you can insert *per* into the problem, it's a good indicator that you're working with a ratio.

Notice that to figure out what value to multiply $\frac{\frac{1}{4}\text{ c}}{1\text{ gal}}$ by in order to form an equivalent ratio with 30 gal in the denominator, we divided 30 gal by 1 gal. This told us that the denominator was 30 times greater, so we needed to multiply the numerator by the *same amount* so that we'd be multiplying by a fraction worth 1, thus forming an equivalent fraction.

Note that sometimes we have to simplify a ratio in order to find the equivalent ratio.

Example: Find x: $\frac{7}{x} = \frac{21}{33}$

Notice that the numerator on the ratio on the right is 3 times the numerator of the ratio of the fraction on the left. That is, $7 \bullet 3 = 21$. So if we divide both the numerator and the denominator of $\frac{21}{33}$ by 3, we'll form an equivalent ratio (we're just simplifying the ratio!) with a 7 in the numerator, thus finding the value of x.

$$\frac{21}{33} = \frac{21 \div 3}{33 \div 3} = \frac{7}{11}$$

x has to equal 11 in order to form an equivalent ratio with a 7 in the numerator.

Keeping Perspective

Ratios and proportions are very common applications of fractions. Want to describe out how much you're making *per* hour if you make \$8.50 per hour? Write a ratio: $\frac{\$8.50}{\text{hr}}$. Want to make a triple batch of a solution to remove wallpaper? Set up a proportion! Remember, math is a useful tool.

1.6 Rates

As we use math outside of a textbook, we end up needing to measure different aspects of the world, such as time, distance, weight, etc. To do this, we use units of measure — agreed upon periods of time, distance, weight, etc., we can use to compare with other units.

For example, we use a second to represent a certain period of time . . . and 60 seconds to represent a minute . . . and 60 minutes to represent an hour.

In this lesson, we're going to take a look at how to work with ratios of units of measure. The great news is that we can treat units of measure just like we would unknowns! So you'll find we'll be applying the same principles we looked at in the last couple of lessons.

But be careful. One of the biggest challenges of working with units of measure in algebra is remembering that **multi-letter units need to be treated as a single entity**. For example, min does *not* mean m times i times n . . . instead, it means minutes.

Understanding and Simplifying Rates

Since we can work with units just like unknowns, we can end up with some interesting units! Speed equals distance divided by time, or $s = \frac{d}{t}$. If your distance is in meters (m) and your time in seconds (s), then when you divide the two, you get $\frac{\text{m}}{\text{s}}$ (i.e., meters *per* second).

Notice that we simplified $\frac{10\text{ m}}{2\text{ s}}$ just like we did other fractions!

Example: A robot travels a distance (d) of 10 meters in 2 seconds (t). What is its speed (s)?

$$s = \frac{d}{t}$$

$$s = \frac{10\text{ m}}{2\text{ s}} = \frac{5\text{ m}}{\text{s}} = 5\frac{\text{m}}{\text{s}}$$

Notice that we wrote the 5 in $5\frac{\text{m}}{\text{s}}$ in front of the $\frac{\text{m}}{\text{s}}$. We could have also written it in the numerator, as $\frac{5\text{ m}}{\text{s}}$.

Both $5\frac{\text{m}}{\text{s}}$ and $\frac{5\text{ m}}{\text{s}}$ mean the *same thing*. Think about it. View the units of measures like you would unknowns. $5\frac{\text{m}}{\text{s}}$ then means 5 *times* $\frac{\text{m}}{\text{s}}$. How would we complete that multiplication? We'd multiply 5 by the numerator, giving us $\frac{5\text{ m}}{\text{s}}$.

Notice also that we were dealing with a *unit* that was a ratio, or **rate**. (A rate is a specific type of ratio. The exact definition for what makes a ratio a rate varies, but the important thing is to know that all rates are ratios — comparisons of 2 quantities using division.)[16] You're used to working with units that are ratios whenever you see speed limits — they give the speed in miles *per* hour, or $\frac{\text{mi}}{\text{hr}}$. In other words, you're looking at how many miles

With ratios or rates, you can read the fraction line as *per*. For example, you can read $\frac{\text{m}}{\text{s}}$ as "meters per second."

you can go in one hour. Likewise, $\frac{\text{m}}{\text{s}}$ means meters *per* second, or how many meters you can go in a second.

Converting Rates

You'll sometimes need to convert rates into other units. For example, say you're driving through Canada and are told the speed limit on a highway is $80\,\frac{\text{km}}{\text{hr}}$. You want to know how many $\frac{\text{mi}}{\text{hr}}$ that is.

Here's the math:

$$80\,\frac{\text{km}}{\text{hr}}\left(\frac{1\text{ mi}}{1.609344\text{ km}}\right) = 80\,\frac{\cancel{\text{km}}}{\text{hr}}\left(\frac{1\text{ mi}}{1.609344\,\cancel{\text{km}}}\right) = \frac{80\ \text{mi}}{1.609344\text{ hr}} \approx 49.710\frac{\text{mi}}{\text{hr}}$$

It's worth noting that, while speed limits are written to show how far you can go in one hour at that speed, they can also be written over different units of time. In fact, in measuring things in science we'll often come up with different rates, such as the one we encountered in an earlier example:

$$\frac{10\text{ m}}{2\text{ s}}$$

This is a speed of 10 meters *per* 2 seconds.

Notice that we wrote the conversion ratio so that the km was in the denominator so it would cancel out with the km in the numerator. We simplified, just as we did with unknowns. Remember, we treat units of measure just as we would unknowns.

Always arrange your conversion ratio so the unit you want to replace will cancel out, leaving your answer in the desired unit.

Now, what if you want to change both the units in a rate? You'll need to multiply it by more than 1 conversion ratio!

Example: Convert $80\frac{\text{km}}{\text{hr}}$ to $\frac{\text{mi}}{s}$.

$$80\frac{\cancel{\text{km}}}{\cancel{\text{hr}}}\left(\frac{1\text{ mi}}{1.609344\,\cancel{\text{km}}}\right)\left(\frac{1\,\cancel{\text{hr}}}{60\,\cancel{\text{min}}}\right)\left(\frac{1\,\cancel{\text{min}}}{60\ s}\right) = \frac{80\text{ mi}}{(1.609344)(60)(60)\ s}$$

$$\approx 0.014\frac{\text{mi}}{s}$$

More with Converting Rates

Let's look at another conversion example. Notice that we rewrite the number as part of the numerator to avoid forgetting to multiply it.

Example: Convert $2\,\frac{\text{m}}{\text{s}}$ to $\frac{\text{m}}{\text{min}}$.

$$\frac{2\text{ m}}{\cancel{\text{s}}}\left(\frac{60\,\cancel{\text{s}}}{1\text{ min}}\right) = \frac{120\text{ m}}{\text{min}} = 120\frac{\text{m}}{\text{min}}$$

Multiplying by a Rate

We can also end up multiplying units together. For example, force is a measure of mass (kg) times the acceleration $\left(\frac{\text{m}}{\text{s}^2}\right)$. So its unit is kg • $\frac{\text{m}}{\text{s}^2}$, which can also be written as $\frac{\text{kg} \bullet \text{m}}{\text{s}^2}$.

If you recall from previous courses, s^2 means s • s. We'll review exponents more thoroughly in the next chapter.

Dividing Units of Measure by Rates

Sometimes when solving problems we end up with crazy units of measure that need simplified. For example, in a future lesson we'll work with a problem that will leave us with this fraction:

$$\frac{40 \text{ mi}}{\frac{35 \text{ mi}}{\text{hr}}}$$

Yikes! How do we simplify this?

Again, just remember to view the units like you would unknowns. We saw in Lesson 1.2 that when we have a fraction divided by a fraction, such as $\frac{1}{5} \div \frac{2}{7}$, which we would write as $\frac{\frac{1}{5}}{\frac{2}{7}}$, we complete the division by inverting and multiplying.

$$\frac{\frac{1}{5}}{\frac{2}{7}} = \frac{1}{5}\left(\frac{7}{2}\right) = \frac{1(7)}{5(2)} = \frac{7}{10}$$

We can do that same thing with units of measure! $\frac{40 \text{ mi}}{\frac{35 \text{ mi}}{\text{hr}}}$ really means 40 mi $\div \frac{35 \text{ mi}}{\text{hr}}$. **We can complete this division by inverting the denominator and multiplying!**

$$\frac{40 \text{ mi}}{\frac{35 \text{ mi}}{\text{hr}}} = 40 \cancel{\text{mi}}\left(\frac{\text{hr}}{35 \cancel{\text{mi}}}\right) \approx 1.143 \text{ hr}$$

Notice that when we did that, the miles (mi) canceled out, leaving us with an answer in hours. We now know that it would take us 1.143 hr to go 40 mi at a speed of $35\frac{\text{mi}}{\text{hr}}$.

Once again, the key to working with units of measure is to view them like you would unknowns.

Keeping Perspective

Hopefully that helped you feel more comfortable with rates. They prove very useful as we apply math. Even representing speed limits uses a rate (miles *per* hour).

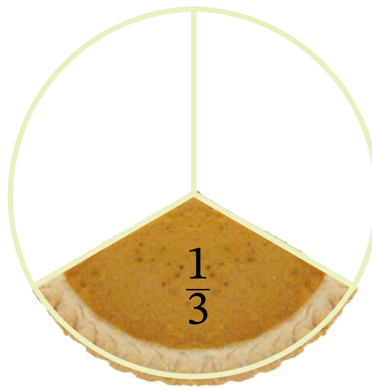
As you review rates and units of measure in general, remember that God knows the measure of all of creation. He is worthy of all our praise.

> *Who hath measured the waters in the hollow of his hand, and meted out heaven with the span, and comprehended the dust of the earth in a measure, and weighed the mountains in scales, and the hills in a balance? (Isaiah 40:12)*

1.7 Adding and Subtracting Fractions

Adding and subtracting fractions is simple if you remember that **the denominators must be the same**. After all, we can only add and subtract divisions by the same quantity!

For example, say you have $\frac{1}{2}$ of a pie plus another $\frac{1}{3}$ of a pie left over from Thanksgiving dinner. What part of a pie altogether do you have? In other words, what is $\frac{1}{2} + \frac{1}{3}$?



Note that we can't just add the numerators, as the divisions are by different amounts. But if we multiply each fraction by a fraction worth 1 that would give a common denominator, then we can add them together. To find a common denominator, just multiply the denominators together. $2 \bullet 3 = 6$, so 6 is a common denominator. Let's multiply both fractions by a fraction worth one to reach that common denominator.

$$\frac{1}{2}\left(\frac{3}{3}\right) + \frac{1}{3}\left(\frac{2}{2}\right) = \frac{3}{6} + \frac{2}{6}$$

Now we can add the numerators, as we're dealing with divisions by the same amount. This would be the same as adding the pieces of the pie together.

$$\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

We have $\frac{5}{6}$ of a pie left.

Adding and Subtracting Rates

Say a bowling ball is traveling at $8\,\frac{\text{m}}{\text{s}}$. If it decreases its speed by $80\,\frac{\text{m}}{\text{min}}$ before it reaches the pins, how fast is it now going?

$$8\frac{\text{m}}{\text{s}} - 80\frac{\text{m}}{\text{min}}$$

Notice that the units are fractions — we're dealing with rates. Remember that we can write the measurements as part of the numerator.

$$\frac{8\text{ m}}{\text{s}} - \frac{80\text{ m}}{\text{min}}$$

Notice that our denominators are different. When adding and subtracting fractions, **the denominators must be the same** (that is, they have to have what we refer to as a **common denominator**). Otherwise you'd be adding and subtracting divisions by different quantities!

Now, we could either solve this problem by converting $\frac{80\text{ m}}{\text{min}}$ to $\frac{\text{m}}{\text{s}}$ or by converting $\frac{8\text{ m}}{\text{s}}$ to $\frac{\text{m}}{\text{min}}$. Since we know that 60 s equals 1 min (see Appendix B: Reference Section for conversion ratios like this one that show us how many of one unit equal another), we'd have these conversions:

$\frac{60\text{ s}}{1\text{ min}}$ is a fraction worth one, as both 60 seconds and 1 minute represent the same amount of time.

Converting to $\frac{\text{m}}{\text{min}}$:	Converting to $\frac{\text{m}}{\text{s}}$:
$\frac{8\text{ m}}{\cancel{\text{s}}}\left(\frac{60\,\cancel{\text{s}}}{1\text{ min}}\right) = \frac{480\text{ m}}{1\text{ min}}$	$\frac{80\text{ m}}{\cancel{\text{min}}}\left(\frac{1\,\cancel{\text{min}}}{60\text{ s}}\right) = \frac{80\text{ m}}{60\text{ s}} \approx \frac{1.333\text{ m}}{\text{s}}$

Now we can subtract the fractions:

$\frac{480\text{ m}}{1\text{ min}} - \frac{80\text{ m}}{\text{min}} = \frac{400\text{ m}}{\text{min}} = 400\frac{\text{m}}{\text{min}}$	$\frac{8\text{ m}}{\text{s}} - \frac{1.333\text{ m}}{\text{s}} = \frac{6.667\text{ m}}{\text{s}} = 6.667\frac{\text{m}}{\text{s}}$

Note: We didn't have to write the 400 to the left of $\frac{\text{m}}{\text{min}}$ or 6.667 to the left of $\frac{\text{m}}{\text{s}}$; an answer of $\frac{400\text{ m}}{\text{min}}$ or $\frac{6.667\text{ m}}{\text{s}}$ would be just as correct. It's just that the numerical value to the left of the units is a little easier to read.

Both $400\frac{\text{m}}{\text{min}}$ and $6.667\frac{\text{m}}{\text{s}}$ represent the ***same rate*** — they are just using different units. One is measuring how many meters are traveled per *minute*, and the other how many meters are traveled per *second*. **Always remember to keep track of your units.** While you do not necessarily have to write them out in each step, be sure to think through units and include them in your answer!

More with Rates

While we're talking about rates, it's important to backtrack and point out that we can only add and subtract **like units**. So not only do the denominators need to be the same to add, but to really complete the addition in the numerator, you'll need to make sure you have like units there as well.

For example, if one robot went $1\frac{\text{ft}}{\text{s}}$ and another went $2\frac{\text{in}}{\text{s}}$, we have the same denominator so we can add the numerators, giving us $\frac{1\text{ ft} + 2\text{ in}}{\text{s}}$. But we can't add 1 ft and 2 in until we first convert to the same unit of measure.

We rewrote 1 ft as 12 in, since both 1 ft and 12 in represent the same length.

$$\frac{1\text{ ft} + 2\text{ in}}{\text{s}} = \frac{12\text{ in} + 2\text{ in}}{\text{s}} = \frac{14\text{ in}}{\text{s}} = 14\frac{\text{in}}{\text{s}}$$

Example: Add $2\frac{\text{ft}}{\text{s}}$ and $5\frac{\text{in}}{\text{min}}$. Give your answer in $\frac{\text{in}}{\text{min}}$.

Notice that to completely add these and get the answer in $\frac{\text{in}}{\text{min}}$, we need to convert the $2\frac{\text{ft}}{\text{s}}$ to $\frac{\text{in}}{\text{min}}$. How do we convert both units in the numerator *and* those in the denominator? The same way we have been, only we'll have to multiply by 2 separate conversion ratios: one to convert each unit.

Conversion ratio to convert seconds to minutes

$$2\frac{\text{ft}}{\text{s}}\left(\frac{60\text{ s}}{1\text{ min}}\right)\left(\frac{12\text{ in}}{1\text{ ft}}\right) = 2\frac{\cancel{\text{ft}}}{\cancel{\text{s}}}\left(\frac{60\ \cancel{\text{s}}}{1\text{ min}}\right)\left(\frac{12\text{ in}}{1\ \cancel{\text{ft}}}\right) = \frac{1{,}440\text{ in}}{\text{min}} = 1{,}440\frac{\text{in}}{\text{min}}$$

Conversion ratio to convert ft to in

Notice that we arranged the conversion ratio however needed to get the appropriate units to cancel out, leaving the requested units.

It's worth noting that we could have done this in 2 steps — converting the feet to inches, and then converting the seconds to minutes. But it saves time to do it all in one step.

Now, we can add this to $5\frac{\text{in}}{\text{min}}$ and get a final answer of $1{,}445\frac{\text{in}}{\text{min}}$.

Dealing with Unknowns

Once again, because of the consistent way God governs all things, we can apply what we know about adding fractions together to unknowns as well! As with known values, the **denominators have to be the same, so we're adding or subtracting divisions by the same quantity.**

Example: Simplify $\frac{a}{2} - \frac{1}{b}$.

Example Meaning: The gear ratio of one gear minus the gear ratio of another gear, where the values *a* and *b* are not known.

$$\frac{ab}{2b} - \frac{2}{2b} = \frac{ab - 2}{2b}$$

Do you see what we did? In order to get $2b$ as the denominator in the first fraction, we had to multiply it by $\frac{b}{b}$. This was multiplying by a value worth 1, which doesn't change the value (the identity property of multiplication).

$$\frac{a}{2}\left(\frac{b}{b}\right) = \frac{ab}{2b}$$

In order to get $2b$ as the denominator in the second fraction, we had to multiply it by $\frac{2}{2}$, which is also worth 1.

$$\frac{1}{b}\left(\frac{2}{2}\right) = \frac{2}{2b}$$

We then combined them by subtracting the second numerator from the first. Since we can't actually complete the subtraction of ab minus 2 as we don't know the value of ab, we leave it written as a subtraction.

$$\frac{ab}{2b} - \frac{2}{2b} = \frac{ab - 2}{2b}$$

If you're not sure what denominator both fractions can be written as, just multiply the denominators together to find a common denominator to use. Notice that the $2b$ denominator we used above was 2 (the denominator of the first fraction) times b (the denominator of the second fraction).

Example: Simplify $2 + \frac{a}{n}$.

Example Meaning: A \$2 base allowance plus whatever additional amount our parents decide to give us divided by the number of siblings they're dividing the additional amount among.

We need to write 2 as a fraction of n. Remember from the last lesson that we can think of 2 as $\frac{2}{1}$.

$$\frac{2}{1} + \frac{a}{n}$$

Notice that we rewrote $1n$ as simply n. We were applying the identity property of multiplication again, knowing that any number times 1 equals itself.

We'll multiply the first fraction by $\frac{n}{n}$, a fraction worth 1 (provided n doesn't equal 0), to get a common denominator. We'll then be able to add the numerators.

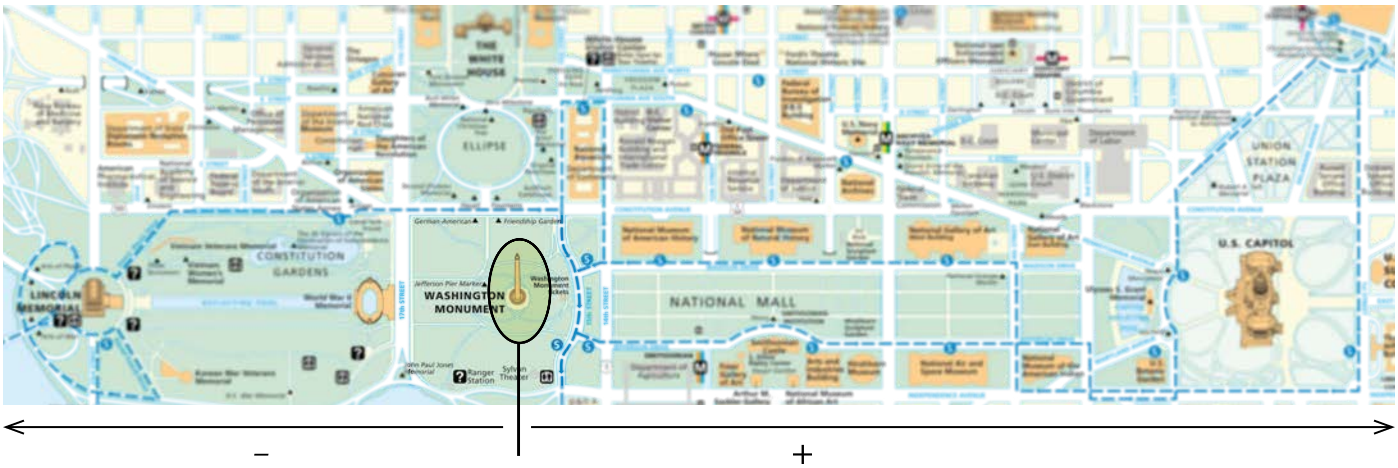
$$\frac{2}{1}\left(\frac{n}{n}\right) + \frac{a}{n} = \frac{2n}{1n} + \frac{a}{n} = \frac{2n}{n} + \frac{a}{n} = \frac{2n + a}{n}$$

Keeping Perspective

Hopefully a lot of what we've been exploring with fractions is review for you. If not, take some extra time to make sure you understand fractions and units of measure, as we use them extensively when we apply math to help us describe God's creation.

1.8 Negative Numbers

As we apply math, we encounter quantities that are the *opposite* of other quantities. We call these **negative numbers**. For example, if you *owe* \$3, you have –\$3, which is *the opposite of* having \$3. Or if you travel in the *opposite* direction of some landmark, you could say you were traveling in the *negative* direction.

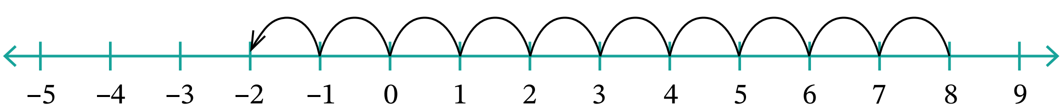


Adding and Subtracting with Negative Numbers

It can be helpful to picture a number line when adding and subtracting negative numbers.

Example: Find 8 – 10.

> If we start at 8, and go 10 spaces to the left, we'll get to 0 after 8 spaces. But we still have 2 to go! Thus our answer will be –2.

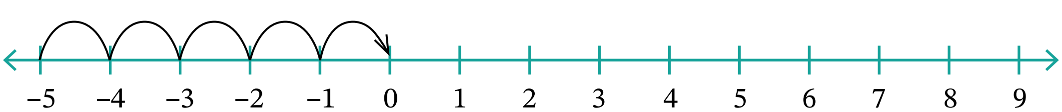


Note that you can view subtraction as an addition of a negative number.

Example: $(5 - x) = (5 + -x)$

It's also worth noting that when you add a number to its opposite, you end up with 0.

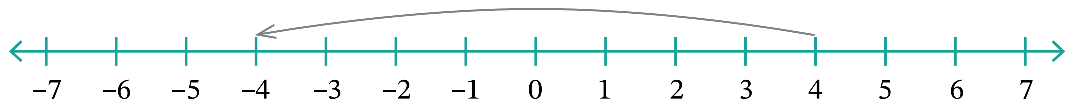
Example: $-5 + 5 = 0$



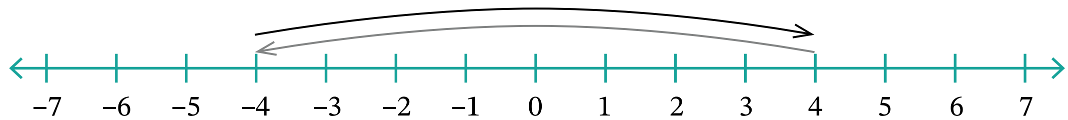
Example: $-a + a = 0$

Multiple Negative Signs

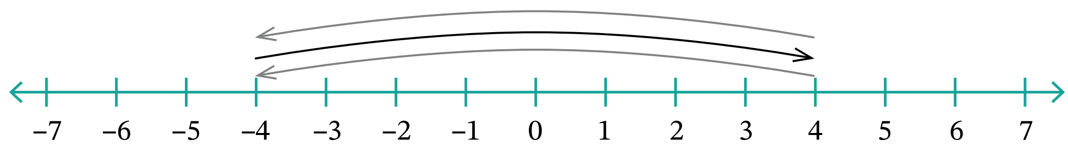
View each negative sign in front of a number as *the opposite of*.



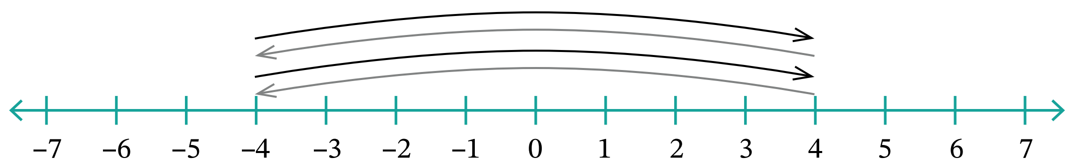
$$-4 = \textit{the opposite of } 4 = -4$$



$$-(-4) = \textit{the opposite of the opposite of } 4 = 4$$



$$-(-(-4)) = \textit{the opposite of the opposite of the opposite of } 4 = -4$$



$$-(-(-(-4))) = \textit{the opposite of the opposite of the opposite of the opposite of } 4 = 4$$

Odd numbers cannot perfectly divide by 2, while even numbers can. See Lesson 3.2 for a more precise definition.

Two negative signs yield a positive answer, three a negative, etc. In fact, you can easily figure out whether the final number is a negative or a positive by saying "negative, positive, negative, positive . . . " as you read off the negative signs. If you end with a negative, the answer is negative; if you end with a positive, the answer is positive. **In short, if there are an odd number of negative signs, the answer will be negative; an even number, the answer will be positive.**

Multiplying and Dividing Negative Numbers

The negative, positive, negative, positive, etc., rule (i.e., odd number of negative signs means negative answer, and even means positive answer) holds true both with negative signs in front of a value *and* when multiplying or dividing.

In fact, one very helpful way of thinking about negative signs in front of a number or unknown is as a multiplication by –1. After all, we can multiply any number by 1 without changing the value. Thus $-a$ equals $-1a$. In fact, many find it useful to think of $-a$ as –1 times a, or $(-1)a$. Viewing it as a multiplication by –1 can help in determining if the negative signs cancel out or not. Remember that each –1 takes the *opposite*, making it negative, positive, etc.

Example: Simplify $-(-(-a))$.

$$-(-(-a)) = (-1)(-1)(-1)a = (-1)a = -a$$

The opposite of the opposite of the opposite of a is $-a$.

Read negative, positive, negative.

Odd number of signs → negative answer

It's not necessary to write out all the −1s . . . they're included just in case that's a helpful way for you to think of the negative signs.

Example: Find $-a$ if a is -3.

Substitute -3 for a: $-(-3) = (-1)(-1)3 = 3$

The opposite of the opposite of 3 is +3!

Even number of signs → positive answer

Example: Simplify $\frac{-4}{-b}$.

$$\frac{-4}{-b} = \frac{(-1)(4)}{(-1)(b)} = \frac{4}{b}$$

Here we have a negative number divided by another negative number. So, our answer ends up with *two* negative signs . . . *the opposite of the opposite* . . . which will be positive.

Even number of signs → positive answer

Another way of thinking about this is that the negative signs cancel out, as one is in the numerator and the other in the denominator.

$$\frac{-4}{-b} = \frac{(\cancel{-1})(4)}{(\cancel{-1})(b)} = \frac{4}{b}$$

Example: Simplify $-\frac{4}{-b}$.

$$-\frac{4}{-b} = (-1)\frac{4}{(-1)(b)} = \frac{4}{b}$$

Even number of signs → positive answer

Another way of thinking about this is that we have a positive number (4) divided by a negative number ($-b$). So, the $\frac{4}{-b}$ part will be negative . . . but then the negative sign in front tells us to take the opposite of that, which would make the final answer positive.

You can also think of there being a negative sign in the numerator and one in the denominator, as $(-1)\frac{4}{(-1)(b)}$ can be rewritten as $\frac{(-1)4}{(-1)(b)}$. The –1s would then cancel out.

Example: Simplify $-\frac{-4}{-b}$.

$$-\frac{-4}{-b} = -1\left(\frac{(\cancel{-1})(4)}{(\cancel{-1})(b)}\right) = -1\left(\frac{4}{b}\right) = -\frac{4}{b}$$

Notice that we had a total of 3 negative signs. Thus, we had *the opposite of the opposite of the opposite*, or a negative answer.

Odd number of signs → negative answer

Another way of thinking about this is that –4 divided by –*b* will be positive, and then the negative sign in front will make that negative again.

Example: Simplify $\frac{3x}{2} - \frac{1}{-2}$.

$$\frac{3x}{2} - \frac{1}{-2} = \frac{3x}{2} + \frac{1}{2} = \frac{3x+1}{2}$$

We could have rewritten $-\frac{1}{-2}$ as $(-1)\left(\frac{1}{(-1)2}\right)$. Notice that then we'd have a –1 in the numerator and the denominator . . . which would cancel out. Thus we end up with positive $\frac{1}{2}$ to add to $\frac{3x}{2}$.

Keeping Perspective

Negative numbers give us a way of describing that a quantity is *the opposite* of something else. We can use them to represent going in *the opposite direction*, owing money (which is *the opposite of* having it — examples would be money you owe on your rent or mortgage), or current flowing in *the opposite direction*. Make sure you're comfortable with working with negative numbers, as they'll come up both throughout this course and in real life.

1.9 Chapter Synopsis

Well, we've reached the end of our first chapter together. In this chapter, we reviewed core concepts that we will use again and again and again and again (did we say that enough times?) as we dig into algebra. All of these concepts rest on the consistent way God governs all things. Our very ability to rely on multiplication to always work the same way, for example, reminds us that God is faithful day after day to govern all things consistently. And He'll be just as faithful to everything else He's said in His Word!

At the end of each chapter, there's typically a review day. But since this chapter was a review chapter, there's no review day. Take just a minute, though, to look over the key skills from this chapter to make sure you're ready for the quiz.

Key Skills for Chapter 1

Understand and apply key properties of addition and multiplication. (Lesson 1.1)

- **Commutative** (order doesn't matter):

  $a + b = b + a$ (for addition)

  $ab = ba$ (for multiplication)

- **Associative** (grouping doesn't matter):

  $(a + b) + c = a + (b + c)$ (for addition)

  $(ab)c = a(bc)$ (for multiplication)

- **Identity** (can multiply by 1 or add 0 without changing value):

  $a + 0 = a$ (for addition)

  $1a = a$ (for multiplication)

Know that any number (except 0) divided by itself equals 1 and that we cannot divide by 0. (Lesson 1.1)

Examples: $\frac{a}{a} = 1$, provided $a \neq 0$

In $\frac{x}{a}$, a cannot equal 0.

Know how to insert known values into a formula. (Lesson 1.1)

Understand the concept of equality and inequality. (Lesson 1.2)

Know various terms (including expressions, equations, and simplify) and conventions (such as that $5x$ means 5 times x and that capital and lowercase letters cannot be used interchangeably — V is a different unknown than v). (Lesson 1.2)

Understand fractions and how to work with them. (See chart; Lessons 1.2–1.4, 1.7)

Understand ratios and proportions and how to work with them. (Lesson 1.5)

- Ratios are comparisons via division; we can write them as a fraction and work with them like fractions.

- A proportion is 2 equal ratios; you can find missing values in a proportion by figuring out what the value would have to be to form an equivalent fraction.

  Example: $\frac{7}{x} = \frac{21}{33}$; $x = 11$

  (The first fraction has to be $\frac{7}{11}$ to form an equivalent fraction with 7 in the numerator — if we multiplied both the numerator and denominator by $\frac{3}{3}$, we'd get $\frac{21}{33}$.)

Understand how to work with units of measure and rates. (Lesson 1.2, Lesson 1.6)

- Multiplying with units (make sure you maintain the units).

  Example: $(7 \text{ kg})(3 \text{ m}) = 21 \text{ kg} \bullet \text{m}$

- Treat units of measure like you would unknowns, only viewing the *entire unit* as a single unknown. For example, treat min (the abbreviation for *minute*) as a single value, not as *m* times *i* times *n*.

- Converting units (multiply by a conversion ratio worth 1) and simplifying units of measure.

  Example: $80\,\frac{\text{km}}{\text{hr}}\left(\frac{1 \text{ mi}}{1.609344 \text{ km}}\right) = 80\,\frac{\cancel{\text{km}}}{\text{hr}}\left(\frac{1 \text{ mi}}{1.609344\ \cancel{\text{km}}}\right) = \frac{80 \text{ mi}}{1.609344 \text{ hr}}$

  $\approx 49.71\,\frac{\text{mi}}{\text{hr}}$

- You can only add and subtract like units.

  Example: $\frac{1 \text{ ft} + 2 \text{ in}}{\text{s}} = \frac{12 \text{ in} + 2 \text{ in}}{\text{s}} = \frac{14 \text{ in}}{\text{s}} = 14\,\frac{\text{in}}{\text{s}}$

Understand negative numbers and how to work with them. (Lesson 1.8)

- **Each negative sign means "the opposite of."**

  Odd number of negative signs = negative answer

  Even number of negative signs = positive answer

- **Addition and Subtraction**

  Add and subtract by thinking about how many times it takes to get to 0 and then how much beyond 0 you have to go.

  Examples: $3 - 4 = -1$

  $$x - (-x) = x + x = 2x$$

  You can view subtraction as an addition of a negative number.

  Example: $(5 - x) = (5 + -x)$

  When you add a number to its opposite, you end up at 0.

  Example: $-a + a = 0$

- **Multiplication and Division**

  When multiplying and dividing or looking at multiple negative signs in front of an expression, an even number of negative signs gives a positive result, and an odd number of negative signs gives a negative result. You can also read off negative, positive, negative, etc.

  Examples: $(-1)(-x) = x$

  $$-(6 - x) = -6 + x$$

  $$\frac{-7x}{2} = -\frac{7x}{2}$$

  $$-\frac{7x}{-2} = \frac{7x}{2}$$

  $$-\frac{-7x}{-2} = -\frac{7x}{2}$$

Fractions

Fractions represent division.

- **Addition**

  Denominators must be the same; then add numerators.

  Example: $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$

- **Subtraction**

  Denominators must be the same; then subtract the second numerator from the first.

  Example: $\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$

- **Multiplication**

  Multiply numerators and denominators.

  Example: $\frac{a}{b}\left(\frac{c}{d}\right) = \frac{ac}{bd}$

  Treat non-fractional quantities as if they had a 1 in the denominator, as dividing by 1 doesn't change the value.

  Example: $x\left(\frac{a}{b}\right) = \frac{x}{1}\left(\frac{a}{b}\right) = \frac{xa}{1b} = \frac{xa}{b}$

- **Division**

  Invert number being divided by and multiply.

  Example: $$\frac{\frac{a}{b}}{\frac{c}{b}} = \frac{a}{b}\left(\frac{b}{c}\right) = \frac{ab}{bc} = \frac{a}{c}$$

- **Reciprocal or Inverse**

  The number that times it equals 1. Flip the numerator and denominator to find.

  Examples: The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

  The reciprocal of 4 is $\frac{1}{4}$.

Chapter 1 Endnotes:

1 Walter W. Sawyer, *Mathematician's Delight* (Harmondsworth Middlesex: Penguin, 1943), p. 10, quoted in James D. Nickel, rev. ed., *Mathematics: Is God Silent?* (Vallecito, CA: Ross House Books, 2001), p. 290.

2 *New Oxford American Dictionary*, 3rd edition (Oxford University Press, 2012), Version 2.2.1 (156) (Apple, 2011), s.v., "algebra," quoted in Loop, *Principles of Mathematics: Book 2*, p. 176.

3 Leonard Euler, *Elements of Algebra, by Leonard Euler, Translated from the French; with the Additions of La Grange, and the Notes of the French Translator* (London: J. Johnson and Co., 1810), https://books.google.com/books?id=hqI-AAAAYAAJ&pg=PR1#v=onepage&q&f=false, p. 270; cited in Katherine A. Loop, *Principles of Mathematics: Book 2* (Green Forest, AR: 2016), p. 176.

4 *New Oxford American Dictionary*, s.v., "formula."

5 When rounding, look at the value to the right of the place to which you want to round. If it is 5 or greater, round up; if it is less than 5, you just round down, or in the case of decimals, leave the place you're rounding to as it is. For example, if rounding 9.578542 to the 3rd decimal, we would look at the 4th decimal place, which is a 5. Since 5 is 5 or greater, we'd round the 3rd decimal place up to the next number, giving us a rounded value of 9.579. But if we had 9.578342 instead, we'd round to 9.578 instead.

6 See Appendix B: Math's Message.

7 Lexico.com, s.v. "convention," https://www.lexico.com/en/definition/convention.

8 Top Row: One of several possible symbols the Egyptians may have used as a form of an equals sign (this particular hieroglyphic means "together" and could have been used to symbolize the results of addition); symbol used by Diophantus (200s); form of modern symbol presented by Recorde (1557); symbol used by Buteo (1559); symbol used by Holzman, better known as Xylander, that several other mathematicians adopted.

9 See Dictionary.com Unabridged, based on *Random House Unabridged Dictionary*, s.v. "kilogram-meter," https://www.dictionary.com/browse/kilogram-meter.

10 *New Oxford American Dictionary*, s.v., "expression."

11 Ibid. s.v., "equation."

12 Florian Cajori, *A History of Mathematics*, 5th rev. ed. (NY: Chelsea Publishing, 1991), p. 123.

13 Lexico.com, s.v. "division," https://www.lexico.com/definition/division.

14 *The American Heritage Dictionary of the English Language*, 1980 New College Edition, s.v. "ratio."

15 See PLTW, Inc., *Engineering Formulas* (n.p., n.d.), p. 6., and Woodgears.ca, "Gear Ratios and Compound Gear Ratios," (n.d.), https://woodgears.ca/gear/ratio.html.

16 See http://mathforum.org/library/drmath/view/58042.html for an exploration of some of the different definitions for "rate" verses "ratio." The author concludes with this: "A rate generally involves a 'something else,' either two different kinds of units (such as distance per time), or just two distinct things measured with the same unit (such as interest money per loaned money)." Doctor Peterson, The Math Forum, "Rate vs. Ratio" The Math Forum @ Drexel, http://mathforum.org/library/drmath/view/58042.html, accessed 10/1/14.